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Forgotten topological index and hyper-Zagreb index of generalized transformation graphs

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Abstract. In this paper, we obtain the expressions for forgotten topological index, hyper-Zagreb index and coindex for generalized transformation graphs G^{ab} and their complements $\overline{G^{ab}}$.

1 Introduction

Let G be a simple, undirected graph with n vertices and m edges. Let $V(G)$ and $E(G)$ be the vertex set and edge set of G respectively. If u and v are adjacent vertices of G, then the edge connecting them will be denoted by uv. The *degree* of a vertex u in G is the number of edges incident to it and is denoted by $d_G(u)$. The *complement* of G, denoted by $\overline{G}$, is a graph having the same vertex set as G, in which two vertices are adjacent if and only if they are not adjacent in G. Thus, the size of $\overline{G}$ is $\binom{n}{2} - m$ and $d_{\overline{G}}(v) = n - 1 - d_G(v)$ holds for all $v \in V(G)$. We refer to [9] for unexplained terminology and notation.

In theoretical chemistry, the physico-chemical properties of chemical compounds are often modeled by means of molecular-graph-based structure-descriptors, which are also referred to as topological indices [8, 12]. The first and second Zagreb indices, respectively, defined

$$M_1(G) = \sum_{u\in V(G)} d_G(u)^2 \text{ and } M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$$

are widely studied degree-based topological indices, that were introduced by Gutman and Trinajstic$'$ [7] in 1972.

Noticing that the contribution of non adjacent vertex pairs to be taken into account when computing the weighted Wiener polynomials of certain composite graphs first and second Zagreb coindex were defined(see [1, 5])

$$\overline{M}_1(G) = \sum_{uv\notin E(G)} [d_G(u) + d_G(v)] \text{ and } \overline{M}_2(G) = \sum_{uv\notin E(G)} d_G(u)d_G(v)$$

respectively.

The vertex-degree-based graph invariant

$$F(G) = \sum_{v\in V(G)} d_G(v)^3 = \sum_{uv\in E(G)} [d_G(u)^2 + d_G(v)^2]$$

was encountered in [7]. Recently there has been some interest to F, called " forgotten topological index" [6].

Shirdel et al.[11] introduced a new Zagreb index of a graph G named "hyper-Zagreb index" and is defined as:

$$HM(G) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))^2.$$

In [3], Basavanagoud et al. corrects some errors of [11] and gave the correct expressions for hyper-Zagreb index of some graph operations.

Recently, Veylaki et al.[13] defined the hyper-Zagreb coindices as

$$\overline{HM}(G) = \sum_{uv\notin E(G)} (d_G(u) + d_G(v))^2.$$

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In [4], Basavanagoud et al. corrects some errors of [13] and gave the correct expressions for hyper-Zagreb coindex of some graph operations.

The following results will be needed for the present considerations.

Proposition 1.1 [4] *Let G be a simple graph on n vertices and m edges. Then* $F(\overline{G}) = n(n-1)^3 - F(G) - 6(n-1)^2 m + 3(n-1)M_1(G)$.

Proposition 1.2 *Let G be a simple graph. Then* $HM(G) = F(G) + 2M_2(G)$.

Proof. By definition of the hyper-Zagreb index, we have

$$\begin{aligned} HM(G) &= \sum_{uv\in E(G)} (d_G(u) + d_G(v))^2 \\ &= \sum_{uv\in E(G)} [d_G(u)^2 + d_G(v)^2 + 2d_G(u)d_G(v)] \\ &= \sum_{uv\in E(G)} (d_G(u)^2 + d_G(v)^2) + 2\sum_{uv\in E(G)} d_G(u)d_G(v) \\ &= F(G) + 2M_2(G). \end{aligned}$$

Proposition 1.3 [4] *Let G be a simple graph. Then* $HM(\overline{G}) = F(\overline{G}) + 2M_2(\overline{G})$.

Proposition 1.4 [4] *Let G be a simple graph on n vertices and m edges. Then* $\overline{HM}(G) = 2\overline{M}_2(G) + (n-1)M_1(G) - F(G)$.

2 Generalized transformation graphs G^{ab}

The semitotal-point graph $T_2(G)$ of a graph G is a graph whose vertex set is $V(T_2(G)) = V(G) \cup E(G)$ and two vertices are adjacent in $T_2(G)$ if and only if (i) they are adjacent vertices of G or (ii) one is a vertex of G and other is an edge of G incident with it. It was introduced by Sampathkumar and Chikkodimath [10]. Recently some new graphical transformations were defined by Basavanagoud et al. [2], which generalizes the concept of semitotal-point graph.

The *generalized transformation graph* G^{ab} is a graph whose vertex set is $V(G) \cup E(G)$, and $\alpha, \beta \in V(G^{ab})$. The vertices α and β are adjacent in G^{ab} if and only if $(*)$ and $(**)$ holds:

$(*)$ $\alpha, \beta \in V(G)$, α, β are adjacent in G if $a = +$ and α, β are not adjacent in G if $a = -$.

$(**)$ $\alpha \in V(G)$ and $\beta \in E(G)$, α, β are incident in G if $b = +$ and α, β are not incident in G if $b = -$.

One can obtain the four graphical transformations of graphs as G^{++}, G^{+-}, G^{-+} and G^{--}. The vertex v_i of G^{ab} corresponding to a vertex v_i of G is referred to as *point vertex* and vertex e_i of G^{ab} corresponding to an edge e_i of G is referred to as *line vertex*.

In [2], Basavanagoud et al. obtained the expressions for first and second Zagreb indices and coindices for generalized transformation graphs G^{ab} and their complements $\overline{G^{ab}}$. Now we obtain the expressions for forgotten topological index, hyper-Zagreb index and coindex for generalized transformation graphs G^{ab} and their complements $\overline{G^{ab}}$.

Proposition 2.1 [2] *Let G be a (n,m)-graph. Then the degree of point and line vertices in G^{ab} are*

1. $d_{G^{++}}(v_i) = 2d_G(v_i)$ and $d_{G^{++}}(e_i) = 2$.
2. $d_{G^{+-}}(v_i) = m$ and $d_{G^{+-}}(e_i) = n-2$.
3. $d_{G^{-+}}(v_i) = n-1$ and $d_{G^{-+}}(e_i) = 2$.
4. $d_{G^{--}}(v_i) = n+m-1-2d_G(v_i)$ and $d_{G^{--}}(e_i) = n-2$.

3 Results

Theorem 3.1 *Let G be an (n,m)-graph. Then* $F(G^{++}) = 8F(G) + 8m$.

Proof. Since G^{++} has $n+m$ vertices and $3m$ edges.

$F(G^{++}) = \sum_{u\in V(G^{++})} d_{G^{++}}(u)^3$

$= \sum_{u\in V(G^{++})\cap V(G)} d_{G^{++}}(u)^3 + \sum_{u\in V(G^{++})\cap E(G)} d_{G^{++}}(u)^3$.

From Proposition 2.1, we have

$F(G^{++}) = \sum_{u\in V(G)} (2d_G(u))^3 + \sum_{u\in E(G)} (2)^3 = 8F(G) + 8m$.

Theorem 3.2 *Let G be an (n,m)-graph. Then* $F(G^{+-}) = nm^3 + m(n-2)^3$.

Proof. Since G^{+-} has $n+m$ vertices and $m(n-1)$ edges.

$F(G^{+-}) = \sum_{u\in V(G^{+-})} d_{G^{+-}}(u)^3$

$= \sum_{u\in V(G^{+-})\cap V(G)} d_{G^{+-}}(u)^3 + \sum_{u\in V(G^{+-})\cap E(G)} d_{G^{+-}}(u)^3$.

In view of Proposition 2.1, we have

$F(G^{+-}) = \sum_{u\in V(G)} (m)^3 + \sum_{u\in E(G)} (n-2)^3 = nm^3 + m(n-2)^3$.

Theorem 3.3 *Let G be (n,m)-graph. Then*

$$F(G^{-+}) = n(n-1)^3 + 8m.$$

Proof. Since G^{-+} has $n+m$ vertices and $m + \frac{1}{2}n(n-1)$ edges.

$F(G^{-+}) = \sum_{u\in V(G^{-+})} d_{G^{-+}}(u)^3$

$= \sum_{u\in V(G^{-+})\cap V(G)} d_{G^{-+}}(u)^3 + \sum_{u\in V(G^{-+})\cap E(G)} d_{G^{-+}}(u)^3$.

From Proposition 2.1, we have

$F(G^{-+}) = \sum_{u\in V(G)} (n-1)^3 + \sum_{u\in E(G)} (2)^3 = n(n-1)^3 + 8m$.

Theorem 3.4 *Let G be (n,m)-graph. Then* $F(G^{--}) = n(n+m-1)^3 - 8F(G) - 12m(n+m-1)^2 + 12(n+m-1)M_1(G) + m(n-2)^3$.

Proof. Since G^{--} has $n+m$ vertices and $\frac{1}{2}n(n-1) + m(n-3)$ edges.

$F(G^{--}) = \sum_{u\in V(G^{--})} d_{G^{--}}(u)^3$

$= \sum_{u\in V(G^{--})\cap V(G)} d_{G^{--}}(u)^3 + \sum_{u\in V(G^{--})\cap E(G)} d_{G^{--}}(u)^3$.

By Proposition 2.1, we have

$F(G^{--}) = \sum_{u\in V(G)} (n+m-1-2d_G(u))^3 + \sum_{u\in E(G)} (n-2)^3$

$F(G^{--}) = n(n+m-1)^3 - 8F(G) - 12m(n+m-1)^2 + 12(n+m-1)M_1(G) + m(n-2)^3$.

Lemma 3.1 [2] *Let G be an (n,m)-graph. Then*

1. $M_1(G^{++}) = 4[m + M_1(G)]$.
2. $M_1(G^{+-}) = nm^2 + m(n-2)^2$.
3. $M_1(G^{-+}) = n(n-1)^2 + 4m$.
4. $M_1(G^{--}) = 4M_1(G) + m(n-2)^2 + (n+m-1)[n(n+m-1) - 8m]$.

Apply Proposition 1.1, Lemma 3.1, from the results of the Theorems 3.1-3.4, we can deduce expressions for forgotten topological index of the complement of generalized transformation graphs $\overline{G^{ab}}$. These are collected in the following.

Theorem 3.5 *Let G be (n,m)-graph. Then*

1. $F(\overline{G^{++}}) = (n+m-1)\{(n+m-1)[(n+m)(n+m-1) - 18m] + 3[4m + 4M_1(G)]\} - 8F(G) - 8m$.

2. $F(\overline{G^{+-}}) = (n+m-1)\{(n+m-1)[(n+m)(n+m-1) - 6m(n-1)] + 3[nm^2 + m(n-2)^2]\} - nm^3 - m(n-2)^3$.
3. $F(\overline{G^{-+}}) = (n+m-1)\{(n+m-1)[(n+m)(n+m-1) - 6m - 3n(n-1)] + 3[n(n-1)^2 + 4m]\} - n(n-1)^3 - 8m$.
4. $F(\overline{G^{--}}) = 8F(G) - m(n-2)^3 + 3m(n-2)^2(n+m-1) + 3(n-n^2-2nm+2m)(n+m-1)^2 + (3n+m)(n+m-1)^3$.

Lemma 3.2 [2] *Let* G *be an* (n,m)*-graph. Then*
1. $M_2(G^{++}) = 4[M_1(G) + M_2(G)]$.
2. $M_2(G^{+-}) = m^3 + m^2(n-2)^2$.
3. $M_2(G^{-+}) = \frac{n-1}{2}[n(n-1)^2 - 2m(n-1) + 8m]$.
4. $M_2(G^{--}) = (n+m-1)\{(n+m-1)[\binom{n}{2} - m] + m(n-2)^2 - 2\overline{M_1}(G)\} + 4\overline{M_2}(G) - 2(n-2)[2m^2 - M_1(G)]$.

Apply Proposition 1.2, Lemma 3.2, from the results of the Theorems 3.1-3.4, we can deduce expressions for hyper-Zagreb index of the generalized transformation graphs G^{ab}. These are collected in the following.

Theorem 3.6 *Let* G *be* (n,m)*-graph. Then*
1. $HM(G^{++}) = 8[F(G) + m + M_1(G) + M_2(G)]$.
2. $HM(G^{+-}) = m^3(n+2) + m(n-2)^2(n+2m-2)$.
3. $HM(G^{-+}) = 2n(n-1)^3 - 2m(n-1)^2 + 8nm$.
4. $HM(G^{--}) = 8\overline{M_2}(G) - 8F(G) + m(n-2)^2 - 4(n-2)(2m^2 - M_1(G)) + [2m(n-2)^2 - 4\overline{M_1}(G) + 12M_1(G)](n+m-1) + (n^2 - n - 14m)(n+m-1)^2 + n(n+m-1)^3$.

Lemma 3.3 [2] *Let* G *be an* (n,m)*-graph. Then*
1. $M_2(\overline{G^{++}}) = 2m[11m + 2n - 3] + 2(2n+2m-5)M_1(G) - 4M_2(G) + (n+m-1)^2[\binom{n+m}{2} - 9m]$.
2. $M_2(\overline{G^{+-}}) = \frac{1}{2}[n^4 + m^4 - 3n^3 + m^3 + 4nm^2 - 2n^2m + 8nm + 3n^2 - 5m^2 - 7m - n]$.
3. $M_2(\overline{G^{-+}}) = \frac{1}{2}[4nm^3 + 3n^2m^2 - 18nm^2 - n^2m + 6nm - 9m^3 + m^4 + 27m^2 - 9m]$.
4. $M_2(\overline{G^{--}}) = 2(n+m-1)\overline{M_1}(G) - 4\overline{M_2}(G) + 2M_1(G)(n+2m-1) + \frac{m}{2}[23m - 8nm - 8n^2 + 16n - 9 + m^2 + m^3]$.

Apply Proposition 1.3, Lemma 3.3, from the results of the Theorem 3.5, we can deduce expressions for the hyper-Zagreb index of the complement of generalized transformation graphs $\overline{G^{ab}}$. These are collected in the following.

Theorem 3.7 *Let* G *be* (n,m)*-graph. Then*
1. $HM(\overline{G^{++}}) = (n+m-1)\{(n+m-1)[2(n+m)(n+m-1) - 36m] + 3[4m + 4M_1(G)]\} - 4[2F(G) + 2m - m(11m + 2n - 3) - (2n+2m-5)M_1(G) + 2M_2(G)]$.
2. $HM(\overline{G^{+-}}) = (n+m-1)\{(n+m-1)[(n+m)(n+m-1) - 6m(n-1)] + 3[nm^2 + m(n-2)^2]\} + n^4 + m^4 - 3n^3 + (1-n)m^3 + 4nm^2 - 2n^2m + 8nm + 3n^2 - 5m^2 - m[7 + (n-2)^3] - n$.

3. $HM(\overline{G^{-+}}) = (n+m-1)\{(n+m-1)[(n+m)(n+m-1) - 6m - 3n(n-1)] + 3[n(n-1)^2 + 4m]\} - n(n-1)^3 + m(6n - n^2 - 17) + m^2[4nm + 3n^2 - 18n - 9m + m^2 + 27]$.
4. $HM(\overline{G^{--}}) = 4[2F(G) - 2\overline{M_2}(G) + (n+2m-1)M_1(G)] + m[23m - 8nm - 8n^2 + 16n + m^2 + m^3 - (n-2)^3 - 9] + [3m(n-2)^2 + 4\overline{M_1}(G)](n+m-1) + 3[n - n^2 - 2nm + 2m](n+m-1)^2 + [3n+m](n+m-1)^3$.

Lemma 3.4 [2] *Let G be an (n,m)-graph. Then*
1. $\overline{M}_2(G^{++}) = 2m(9m-1) - 6M_1(G) - 4M_2(G)$.
2. $\overline{M}_2(G^{+-}) = \frac{m}{2}[m(2n^2 - 2m - 4 - n) - (n-2)^2]$.
3. $\overline{M}_2(G^{-+}) = 2[\binom{n}{2} + m]^2 - (n-1)[n\binom{n}{2} + 5m - mn] - 2m$.
4. $\overline{M}_2(G^{--}) = 2(n+m-1)\overline{M_1}(G) - 4\overline{M_2}(G) - 2(n-1)M_1(G) + \frac{1}{2}[m^2n^2 + 16m^2 - 4m^2n - 3mn^2 + 4nm - 2m + 2m^3]$.

Apply Proposition 1.4, Lemmas 3.4, 3.1, from the results of the Theorems 3.1-3.4, we can deduce expressions for hyper-Zagreb coindex of the generalized transformation graphs G^{ab}. These are collected in the following.

Theorem 3.8 *Let G be (n,m)-graph. Then*
1. $\overline{HM}(G^{++}) = 4m(n + 10m - 4) + 4M_1(G)(n+m-4) - 8[M_2(G) + F(G)]$.
2. $\overline{HM}(G^{+-}) = m[m(2n^2 - 2m - 4 - n) - (n-2)^2 - nm^2 - (n-2)^3] + (n+m-1)[nm^2 + m(n-2)^2]$.
3. $\overline{HM}(G^{-+}) = 4[\binom{n}{2} + m]^2 - 2(n-1)[n\binom{n}{2} + 5m - mn] - 12m + (n+m-1)[n(n-1)^2 + 4m] - n(n-1)^3$.
4. $\overline{HM}(G^{--}) = 4\big[2F(G) - 2\overline{M_2}(G) - (n-1)M_1(G)\big] + m^2n^2 + 16m^2 - 4m^2n - 3mn^2 + 4nm - 2m + 2m^3 - m(n-2)^3 + [4\overline{M_1}(G) - 8M_1(G) + m(n-2)^2](n+m-1) + 4m(n+m-1)^2$.

Lemma 3.5 [2] *Let G be an (n,m)-graph. Then*
1. $\overline{M}_2(\overline{G^{++}}) = 4M_2(G) - 4(n+m-2)M_1(G) + m(3m^2 - 10m + 3n^2 + 6nm - 10n + 7)$.
2. $\overline{M}_2(\overline{G^{+-}}) = m[n^2m + 2n^2 - 3mn + 2m - 5n + 3]$.
3. $\overline{M}_2(\overline{G^{-+}}) = \frac{1}{2}[(n+m)(n+m-1) - 2m - n(n-1)^2] - \frac{1}{2}[4nm^3 + 3n^2m^2 - 15nm^2 - 8m^3 + m^4 + 21m^2]$.
4. $\overline{M}_2(\overline{G^{--}}) = 4\overline{M_2}(G) - 2(n+m-1)\overline{M_1}(G) - 2(n+2m)M_1(G) + \frac{1}{2}[8m^3 + 8nm^2 + 8n^2m - 16nm + 8m]$.

Lemma 3.6 [2] *Let G be (n,m)-graph. Then*
1. $M_1(\overline{G^{++}}) = 4[M_1(G) + m] + (n+m-1)[(n+m)(n+m-1) - 12m]$.
2. $M_1(\overline{G^{+-}}) = n(n-1)^2 + m(m+1)^2$.
3. $M_1(\overline{G^{-+}}) = m^3 + 3nm^2 + n^2m - 6m^2 - 6nm + 9m$.
4. $M_1(\overline{G^{--}}) = 4M_1(G) + m^3 + 2m^2 + m$.

Apply Proposition 1.4, Lemmas 3.5, 3.6, from the results of the Theorem 3.5, we can deduce expressions for hyper-Zagreb coindex of the complement of generalized transformation graphs $\overline{G^{ab}}$. These are collected in the following.

Theorem 3.9 *Let G be (n,m)-graph. Then*

1. $\overline{HM}(\overline{G^{++}}) = 8[M_2(G) - (n+m-2)M_1(G) + F(G) + m] + 2m(3m^2 - 10m + 3n^2 + 6nm - 10n + 7) + (n+m-1)[6m(n+m-1) - 8m - 8M_1(G)]$.
2. $\overline{HM}(\overline{G^{+-}}) = 2m(n^2m + 2n^2 - 3nm + 2m - 5n + 3) + nm^3 + m(n-2)^3 + (n+m-1)\{n(n-1)^2 + m(m+1)^2 - 3[nm^2 + m(n-2)^2] - (n+m-1)[(n+m)(n+m-1) - 6m(n-1)]\}$.
3. $\overline{HM}(\overline{G^{-+}}) = n(n-1)^2(n-2) + 6m - 4nm^3 - 3n^2m^2 + 15nm^2 + 8m^3 - m^4 - 21m^2 + (n+m-1)\{n - 2m + m^3 + 3nm^2 + n^2m - 6m^2 - 6nm - 3n(n-1)^2\} - (n+m-1)^2[(n+m)(n+m-1) - 6m - 3n(n-1)]$.
4. $\overline{HM}(\overline{G^{--}}) = 4[2\overline{M_2}(G) - (n+2m)M_1(G) - 2F(G)] + 8m[m^2 + nm + n^2 - 2n + 1] + m(n-2)^3 + [4M_1(G) + m^3 + 2m^2 + m - 4\overline{M_1}(G) - 3m(n-2)^2](n+m-1) - 3[n - n^2 - 2nm + 2m](n+m-1)^2 - (3n+m)(n+m-1)^3$.

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