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(τ_i, τ_j) pgprw closed and open sets in bitopological spaces

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Abstract
In this paper, we introduce and investigate the concept of (τ_i,τ_j)-pgprw-closed sets which are introduced in a bitopological spaces in analogy with pgprw-closed sets in topological spaces. From now, τ −p-cl(A) denotes the pre-closure of A of a relative to a topology τ and during this process some of their properties are obtained.

Keywords: pgprw-closedsets, pgprw-open sets, pgprw-closed set, (τ_i, τ_j) pgprw-open set
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1. Introduction

The triple (X,τ_1,τ_2), where X is a set and τ_1,τ_2 are topologies on X is called a bitopological space. Kelly [1] initiated the systematic study of such spaces in 1963. Following the work of kelly on the bitopological spaces, various authors like Arya and Nour [2], Di maio and Noiri [3], Fukutate [4], Nagaveni [5], Maki,Sundaram and Balachandran [6], Shiek john [7],Sampath kumar [8], Patty [9], Arockiarani [10], Gnanambal [11], Reily [12], Rajamani and Viswanthan [13] and Popa [14] have turned their attention to the various concepts of topology by considering bitopological spaces instead of topological spaces.

2. Preliminaries

Let i,jϵ{1,2} be fixed integers, in a bitopological spaces (X,τ_1,τ_2) a subset A of (X,τ_1,τ_2) is said to be

(i) (i,j)-g-closed[4] if τ_j-cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is τ_i.
(ii) (i,j)-rg-closed[10] if τ_j-cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is regular open in τ_i.
(iii) (i,j)gpr-closed[11] if τ_j-cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is regular open in τ_i.
(iv) (i,j)wg-closed[5] if τ_j-cl(A)(τ_i-int(A)) $\subseteq$ U whenever A $\subseteq$ U and U is τ_i.
(v) (i,j)W-closed[7] if τ_j-cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in τ_i.
(vi) (i,j)gp-closed[15] if τ_j-pcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is τ_i.
(vii) Let (X,P,L) be a bitopological space. We say that A⊆ X is p-semi-open [16] w.r.t L iff there exists a P open set O⊆ X s.t O⊆ A⊆ L-cl(o).
Similarly A⊆ X is L-semi open w.r.t P iff there exists L open set O⊆ X s.t O⊆ A⊆ p-cl(o). Then A is said to be semi-open iff it is both P semi-open w.r.t L and L is semi-open w.r.t P.
(viii) In (X,τ_1,τ_2) A⊆ X is said to be (i,j) pre-open (i,j) [17] p.o iff A⊆, τ_i int (τ_j –cl(A)),
(ix) i,j=1,2; i≠j.
(x) (i,j) rgα-closed [18]if τ_j-αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is rα-open in τ_i.
(xi) Regular open set [19] if A = int(clA)) and a regular closed set if A = cl(int(A)).
(xii) Regular generalized α-closed set(briefly,rgα-closed) [20]if αcl(A)⊆UwheneverA⊆Uand U is regular α-open in X.
(xiii) For a subset A of (X, τ), pgprw-closure of A is denoted by pgprw-cl(A) and defined as pgprw-cl(A)= ∩{G: A⊆G, G is pgprw–closed in (X, τ)} or ∩{G:A⊆G, G ϵ pgprw-C(X)}.

3.0 ($\tau_{i,}$ τ_j)pgprw closed sets and their basic properties.

In this section, we introduce the investigate the concept of ($\tau_{i,}$ τ_j)pgprw-closed sets which are introduced in a bitopological spaces in analogy with pgprw-closed sets in topological spaces. From

Now on, τ $_{-}$pcl(A) denotes the closure of A relative to a topology τ.

3.1 Definition: Let i,jϵ{1,2} be fixed integers,in a bitopological spaces (X,τ $_{1,}\tau_2$), a subset A⊆ X

is said to be ($\tau_{i,}$ τ_j) pgprw-closed set if τ_j-pcl(A) $\subseteq$ G whenever A $\subseteq$ G and G ϵ rgα-open (X, τ_i)

We denote the family of all (i,j) pgprw-closed in a bitoplogical spaces (X,τ $_{1,}\tau_2$) by D_{pgprw} ($\tau_{i,}$ τ_j).

3.2 Remark: By setting τ $_{1=}$ τ_2 in definition 3.1, an (i,j) pgprw-closed set reduces to a pgprw-closed in X.

First we prove that the class of (i,j) pgprw-closed sets properly lies between the class of

(i,j) pre-closed sets and the class of (i,j) gpr-closed sets.

3.3 Theorem: If A is (i,j) pre-closed set subset of (X,τ $_{1,}\tau_2$), then A is (i,j) pgprw-closed.

Proof: Let A be a (i,j) pre closed subset of (X,τ $_{1,}\tau_2$).Let G ϵ rgα-open (X, τ_i) be s.t A $\subseteq$ G. Since A is pre-closed we have τ_j-pcl(A) = A, which implies τ_j-pcl(A) $\subseteq$ G. Therefore A is (i,j) pgprw-closed.

The converse of this theorem need not be true as seen from the following example.

3.4 Example: Let X={a,b,c}, τ_1= {X,∅,{a,b}}and τ_2 = { X,∅,{a},{b},{a,c}}.

Then the subset {a,b} is (1,2) is pgprw-closed set but not (1,2) pre-closed set in the bitopological spaces (X,τ $_{1,}\tau_2$).

3.5 Theorem: If A is a (i,j)- pgprw-closed subset of (X,τ $_{1,}\tau_2$), then A is (i,j) gpr-closed.

Proof: Let A be a (i,j) pgprw-closed subset of (X,τ $_{1,}\tau_2$). Let G ϵ regular-open (X,τ_j) be

s.t A $\subseteq$ G since RO(X,τ_j)⊆ rgα-open (X, τ_i) We have G ϵ rgα-open (X, τ_i) then by hypothesis

τ_j-pcl(A) $\subseteq$ G, also τ_j-pcl(A) $\subseteq$$\tau_j$-pcl(A) which implies τ_j-pcl(A) $\subseteq$ G, Therefore A is (i,j) gpr-closed.

The converse of the theorem need not be true as seen from the following example.

3.6 Example: Let X={a,b,c}, τ_1= {X,∅,{a},{b},{a,b}}and τ_2 = { X,∅,{a}}.

Then {a,b},{a,c} are (1,2) is gpr-closed set but not (1,2) pgprw-closed sets in the bitopological spaces.

3.7 Theorem: If A is τ_j**-** closed subset of a bitopological space (X,τ $_{1,}\tau_2$), then the set A is (i,j)

Pgprw-closed.

Proof: Let, G ϵ rgα-open (X, τ_i) be s.t A $\subseteq$ G then by hypothesis τ_j-pcl(A) = A. Which implies

τ_j-pcl(A) $\subseteq$ G therefore A is (i,j) pgprw-closed.

The converse of this theorem need not be true as seen from the followin example.

3.8 Example: Let X={a,b,c}, τ_1= {X,∅,{a}and τ_2 = { X,∅,{a},{a,b}}.

Then the subset {b}is(1,2) is pgprw-closed set but not τ_2 closed sets in the bitopological space

(X,τ $_{1,}\tau_2$).

3.9 Remark: τ_j**-** w-closed set and (i,j) pgprw-closed set are independent as seen from the following examples.

3.10 Example: Let X={a,b,c}, τ_1= {X,∅,{a},{b},{a,b}}and τ_2= { X,∅,{a},{a,b}}.

Then the subset {c}is(1,2) is pgprw-closed set but not τ_2 w-closed sets in the bitopological space

(X,τ $_{1,}\tau_2$).

3.11 Example: Let X={a,b,c}, τ_1= {X,∅,{a},{b},{a,b}}and τ_2= { X,∅,{a},{a,b}}.

Then the subset {a,c}is(1,2) is τ_2 –w-closed set but not (1,2)pgprw-closed set in the bitopological space (X,τ $_{1,}\tau_2$).

3.12 Remark: τ_j**-** pgprw-closed set and (i,j) semi-closed set are independent as seen from the following examples.

3.13 Example: Let X={a,b,c}, τ_1= {X,∅,{a},{a,b}}and τ_2 = { X,∅,{a,b}}.

Then the subset {b}is(1,2) τ_2 is pgprw-closed set but not (1,2) semi-closed set in the bitopological space (X,τ $_{1,}\tau_2$).

3.14 Example: Let X={a,b,c},τ_1= {X,∅,{a}}and τ_2 = { X,∅,{a},{b},{a,b}}.

Then the subset {b}is semi-closed set but not τ_2 (1,2) pgprw-closed set in (X,τ $_{1,}\tau_2$).

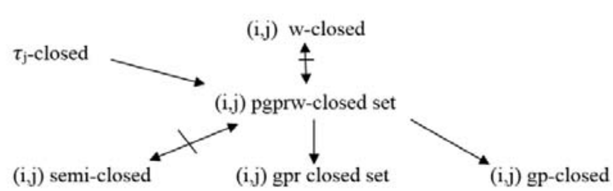
3.15 Example: Let X={a,b,c}, τ_1= {X,∅,{a}}and τ_2 = { X,∅,{a},{a,b}}.

Then {a,c}is(1,2) gp-closed set but not (1,2) pgprw-closed set in the bitopological space (X,τ $_{1,}\tau_2$).

3.16 Remark: From the above discussion and known results we have the following implication

A⟶ B means A implies B but not convesely

A⟶ B means A and B are independent of each other



3.17 Theorem: If A,Bϵ D_{pgprw} (i,j) then A∪Bϵ D_{pgprw} (i,j).

Proof: Let Gϵ rgα-open (X, τ_i)be s.t A∪B⊆ G, then A⊆G and B⊆G since A,Bϵ D_{pgprw} (i,j).

We have τ_j-pcl(A) ⊆G and τ_j-pcl(B) ⊆G that is τ_{j}-pcl(A) ∪ τ_j-pcl(B) ⊆G also

τ_j-pcl(A) ∪ τ_j-pcl(B) = τ_j-pcl(A∪B) and so τ_j-pcl(A∪B) ⊆G Therefore A∪Bϵ D_{pgprw} (i,j).

3.18 Remark: The intersection of two (i,j) pgprw-closed sets in generally a (i,j) pgprw-closed set as seen from the following example.

3.19 Example
Let $X=\{a,b,c,d\},\tau_1=\{X,\emptyset,\{a\},\{b\},\{a,b\},\{a,b,c\}\}$ and $\tau_2 = \{Y,\emptyset,\{a\},\{c,d\},\{a,c,d\}\}$.
Then the subsets {a,c,d} and {c,d} are (1,2) pgprw-closed sets then $\{a,c,d\} \cap \{c,d\}= \{c,d\}$
is a (1,2) pgprw-closed set in the bitopological space (X,τ_1,τ_2).

3.20 Remark : The family D_{pgprw} (1,2) is generally not equal to the family D_{pgprw} (2,1) as seen from the following example.

3.21 Example: Let $X = \{a,b,c\}$, $\tau_1= \{X,\emptyset,\{a\},\{b\},\{a,b\}\}$ and $\tau_2= \{Y,\emptyset,\{a,b\}\}$ then
$D_{pgprw}(1,2) = \{X,\emptyset,\{c\},\{b,c\},\{a,c\}\}$ and $D_{pgprw}(2,1) = \{X,\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\}\}$
Therefore $D_{pgprw}(1,2) \neq D_{pgprw}(2,1)$.

3.22 Theorem : If $\tau_1 \subseteq \tau_2$ and rgα-open $(X, \tau_1) \subseteq$ rgα-open (X, τ_2) in (X,τ_1,τ_2) then
$D_{pgprw}(\tau_2,\tau_1) \subseteq D_{pgprw}(\tau_1,\tau_2)$.

Proof: Let $A\epsilon D_{pgprw}(\tau_2,\tau_1)$ that is a (2,1)-pgprw-closed set. To prove that $A\epsilon D_{pgprw}(\tau_1,\tau_2)$.
$G\epsilon$ rgα-open (X, τ_1)be s.t $A\subseteq G$ since rgα-open $(X, \tau_1) \subseteq$ rgα-open(X, τ_2), we have
$G\epsilon$ rgα-open (X, τ_2)as A is a (2,1) pgprw-closed set, we have τ_1-pcl(A) $\subseteq$G since $\tau_1 \subseteq \tau_2$ we have
τ_2-pcl(A) $\subseteq$ τ_1-pcl(A) and it follows that τ_2-pcl(A) $\subseteq$ G. Hence A is (1,2) pgprw-closed that is
$A\epsilon D_{pgprw}(\tau_1,\tau_2)$ therefore $D_{pgprw}(\tau_2,\tau_1) \subseteq D_{pgprw}(\tau_1,\tau_2)$.

3.23Theorem: Let i,j be fixed integers of {1,2} for each x of (X,τ_1,τ_2) {x} is rgα-open (X, τ_i)
or $\{x\}^c$ is (i,j) pgprw-closed.

Proof: Suppose {x} is not rgα-open(X, τ_i)then $\{x\}^c$ is not rgα-open (X, τ_i). Now rgα-open (X, τ_i).
Containing $\{x\}^c$ is X alone Also τ_j-pcl(A)($\{x\}^c$) $\subseteq$ X. Hence $\{x\}^c$ is (i,j) pgprw-closed.

3.24 Theorem: If A is (i,j)pgprw-closed, then τ_j-pcl(A)-A contains no non-empty τ_i-rgα-open.

Proof: Let A be a (i,j)-pgprw-closed set. Suppose F is a non-empty τ_i rgα-open contained in
τ_j-pcl(A) – A. Now $F\subseteq$ X-A which implies $A\subseteq F^c$ also F^c is a τ_i-rgα-open. Since A is a (i,j)-pgprw-closed set, we have τ_j-pcl(A) $\subseteq F^c$. Consequently $F\subseteq \tau_j\text{-pcl}(A)\cap(\tau_j\text{-pcl}(A)^c = \emptyset$ which is a contradiction. Hence τ_j-pcl(A)-A contains no non-empty τ_i rgα-open.
The converse of this theorem does not holds as seen from the following example.

3.25Example : Let $X=\{a,b,c,d\}$, $\tau_1= \{X,\emptyset,\{a\},\{c,d\},\{a,c,d\}\}$ and
$\tau_2 = \{X,\emptyset,\{a\},\{b\},\{a,b\},\{a,b,c\}\}$then the set A= {a,d} then τ_j-pcl(A)-A= {a,c,d}-{a,d}={c}
does not contain any non-empty τ_i-rgα-open set but A is not a (1,2)pgprw-closed set in the bitopological space (X,τ_1,τ_2).

3.26Corollary: If A is (i,j) pgprw-closed in (X,τ_1,τ_2), then A is τ_j-closed iff τ_j-pcl(A)-A is a
τ_i-rgα-open set.

Proof: suppose A is τ_j-closed then τ_j-pcl(A) = A and so τ_j-pcl(A) – A = $\emptyset$ which is a τ_i-rgα-open set. Conversely, suppose τ_j-pcl(A) – A is a τ_i rgα-open. Since A is (i,j) pgprw-closed by thm 3.24
τ_j-pcl(A) – A does not contain any non-empty τ_i-rgα-open set. Therefore τ_j-pcl(A) – A = $\emptyset$. That is τ_j-pcl(A) = A and hence A is τ_j-closed.

3.27Theorem: In a bitopological space (X,τ_1,τ_2), rgα-open set (X, τ_i).$\subseteq \{F\subseteq X:F^c\epsilon\tau_j\}$ iff every subset of (X,τ_1,τ_2) is a (i,j) pgprw-closed.

Proof: Suppose that rgα-open set $(X, \tau_i)\subseteq \{F\subseteq X:F^c\epsilon\tau_j\}$. Let A be any subset of x. Let $G\epsilon$ rgα-open (X,τ_i). be s.t $A \subseteq G$.
Then τ_j-pcl(G) = G. Also τ_j-pcl(A) $\subseteq$ τ_j-pcl(A) = G.
That is τ_j-pcl(A) $\subseteq$ G. Therefore A is a (i,j) pgprw-closed.
Conversely, Suppose that every subset of (X,τ_1,τ_2) is a (i,j)pgprw-closed set .
Let $G\epsilon$ rgα-open (X,τ_i). Since $G \subseteq G$ and G is (i,j)pgprw-closed set, we have τ_j-pcl(G) $\subseteq$ G.
Thus τ_j-pcl(G) $\subseteq$ G and so G is τ_j-closed. That is $G\epsilon\{F\subseteq X: Fc\epsilon\tau j\}$
hence rgα-open $(X, \tau_i)\subseteq F\subseteq X:F^c\epsilon\tau_j\}$.

3.28Theorem: Let A be a (i,j) pgprw-closed subset of a bitopological space (X,τ_1,τ_2). If A is
τ_i-rgα-open, then A is τ_j-closed.

Proof: Let A be rgα-open (X, τ_i). Now, $A \subseteq A$. Then by hypothesis τ_j-pcl(A) $\subseteq$ A. Therefore
τ_j-pcl(A) = A. That is A is τ_j-closed.

3.29Theorem: If A is a (i,j) pgprw-closed set and $\tau_i \subseteq$ rgα-open $(X, \tau i)$ then τ_j-pcl$\{(x)\}\cap A\neq \emptyset$,
For each $x\epsilon$ τ_j-pcl (A).

Proof: Let A be (i,j) pgprw-closed and $\tau_i \subseteq$ rgα-open $(X, \tau i)$ suppose τ_j-pcl$\{(x)\}\cap A = \emptyset$,
For some $x\epsilon$ τ_j-pcl (A) then $A\subseteq$ $(\tau_i\text{-pcl}\{(x)\}^C$. Now $(\tau_i\text{-pcl}\{(x)\}^C\epsilon\ \tau_i \subseteq$ rgα-open (X, τ_i).
By hypothesis that is $(\tau_i\text{-pcl}\{(x)\}^C$ is τ_i -rgα-open; since A is (i,j)pgprw-closed,
we have τ_j-pcl (A) $\subseteq$ $(\tau_i\text{-pcl}\{(x)\}^C$. This shows that $X \notin \tau_j$-pcl (A) this contradicts the assumption.

3.30Theorem: If A is a (i,j) pgprw-closed set and $A \subseteq B\subseteq \tau_j$-pcl (A), then B is (i,j) pgprw-closed.

Proof: Let G be a τ_i-rgα-open set s.t $B\subseteq G$ as A is a (i,j)pgprw-closed set and $A\subseteq G$, we have
τ_j-pcl (A) $\subseteq$G. Now $B\subseteq \tau_j$-pcl (A) implies τ_j-pcl (B) $\subseteq$ τ_j-pcl $\{\tau_j$-pcl (A)$\}$ = τ_j-pcl (A) $\subseteq$G.
Thus τ_j-pcl (B) $\subseteq$ G. Therefore B is a (i,j) pgprw-closed set.

3.31Theorem : Let $A\subseteq Y\subseteq X$ and suppose that A is (i,j)pgprw-closed in (X,τ_1,τ_2) then A is (i,j)
Pgprw-closed in (X,τ_1,τ_2) then A is (i,j) pgprw-closed relative to Y provided Y is a
τ_i-regular-open set.

Proof: Let, $\tau_{i\text{-}Y}$ be the restriction of τ_i to Y. Let G be $\tau_{i\text{-}Y}$ rgα-open s.t $A \subseteq G$. Since $A \subseteq Y\subseteq X$

and Y is τ_i-regular-open set since w.k.t A $\subseteq$Y$\subseteq$X where X is a t.s and Y is an open subspace of X.

If Aϵ rgα-open(X) then Aϵ rgα-open(X) then G is τ_i -rgα-open. Since A is (i,j) pgprw-closed τ_j-pcl (A) $\subseteq$ G.That is Y∩ τ_j-pcl (A) $\subseteq$ Y∩G =G. Also Y∩ τ_j-pcl (A) = $\tau_{j\text{-}y}$ -pcl (A).

Thus $\tau_{j\text{-}y}$ -pcl (A) $\subseteq$ G.Hence A is (i,j) pgprw-closed relatively to Y.

3.32Theorem: In a bitopological space (X,τ_1,τ_2) if rgα-open (X,τ_i).) = {X,∅}, then every subset of (X,τ_1,τ_2) is (i,j) pgprw-closed.

Proof: Let rgα-open (X,τ_i).) = {X,∅}, in a bitopological space (X,τ_1,τ_2).Let A be any subset of X. To prove that A is an (i,j)-pgprw-closed. Suppose A= ∅ then A is (i,j)-pgprw-closed.

Suppose $A \neq \emptyset$, then X is the only τ_i rgα-open and τ_j-pcl (A) $\subseteq$ X. Hence A is a (i,j) pgprw-closed set.

The converse of the above theorem need not be true in general as seen from the following example.

3.33Example: Let X = {a,b,c}, τ_1= {X,∅,{a},{b},{a,c}}and τ_2 = { Y,∅,{a,b}} then every subset of X is a (1,2) is a pgprw-closed set but rgα-open (X, τi) = {X,∅,{a},{b},{c}}.

3.34Example: If A is τ_i-open and (i,j) g-closed Then A is (i,j)pgprw-closed.

Proof : Let G be a τ_i- rgα-open set s.t A$\subseteq$G, now A$\subseteq$A,A is τ_i-open and (i,j)-g-closed.

We have τ_j-pcl (A) $\subseteq$ A. that is τ_j-pcl (A) $\subseteq$ G. Therefore A is (i,j) pgprw-closed.

3.35Theorem: suppose that B$\subseteq$A$\subseteq$ X. B is a (i,j) pgprw-closed set relative to A and that A is both τ_i-clopen and τ_j-closed. Then B is (i,j)-pgprw closed set in (X,τ_1,τ_2).

Proof: Let $\tau_{i\text{-}A}$ be the restriction of τ_i to A. Let B$\subseteq$G and G be τ_i-rgα-open but it is given that B$\subseteq$A$\subseteq$ X. Therefore B$\subseteq$G and B$\subseteq$A, which implies B$\subseteq$ A ∩ G. Now we show that A ∩ G is $\tau_{i\text{-}A}$ rgα-open since A is τ_i-open and G is τ_i is semi-open, A ∩ G is τ_i is semi-open. Since A is τ_i- closed and G is τ_i-semi closed. A ∩ G is τ_i is semi-closed.Thus A ∩ G is both τ_i is semi-open and τ_i is semi-closed and hence A ∩ G is τ_i rgα-open set. Since A ∩ G $\subseteq$ A $\subseteq$ X. Since w.k.t A $\subseteq$Y$\subseteq$X, where X is a topological space and Y is an open subspace of X. If Aϵrgα(X),then Aϵ rgα(X)implies A ∩ G is $\tau_{i\text{-}A}$ - rgα-open set.Since B is a (i,j) pgprw-closed relative to A

$\tau_{j\text{-}A}$-pcl(B)$\subseteq$ A ∩ G.........(i) But $\tau_{j\text{-}A}$-pcl(B) = A∩ τ_j-pcl(B) ....(ii) from (i) and (ii), it follows that A∩ τ_j-pcl(B) $\subseteq$ A ∩ G. Consquently A∩ τ_j-pcl(B)$\subseteq$ G. Since A is τ_j- closed. τ_j-pcl(A) = A and

τ_j-pcl(B)$\subseteq$$\tau_j$-pcl(A); we have A∩ τ_j-pcl(B) = τ_j-pcl(B). Thus τ_j-pcl(B) $\subseteq$ G and hence B is (i,j)pgprw-closed Set in (X,τ_1,τ_2).

3.36 Definition: Let i,jϵ{1,2} be fixed integers, in a bitopological space (X,τ_1,τ_2), a subset A$\subseteq$ X,

is said to be (τ_i,τ_j)pgprw-open if A^c is (i,j) pgprw-closed.

3.37Theorem : In a bitopological spaces (X,τ_1,τ_2)we have the following

(i) Every (i,j) pre-open set is (i,j) pgprw-open but not conversely.

(ii) Every τ_j -open set is (i,j) pgprw-open but not conversely.

(iii) Every τ_j –pgprw open set is (i,j) gpr-open but not conversely.

Proof: The proof follows from the theorems 3.3, 3.5, & 3.7.

3.38 Theorem : If A and B are (i,j) pgprw-open sets, then A ∩ B is (i,j) pgprw-open.

Proof: The proof follows from the theorem 3.17

3.39 Remark : The union of two (i,j) pgprw-open sets is generally an (i,j) pgprw-open set as seen from the following example.

3.40 Example

Let X={a,b,c,d}, τ_1={X,∅,{a},{b,{a,b},{a,b,c}}and

τ_2 = { Y,∅,{a},{c,d},{a,c,d}},then the subsets {b} and {a,b} are {1,2} pgprw-open sets and

{b}∪ {a,b}= {a,b} is a (1,2) pgprw-open set in the bitopological space (X,τ_1,τ_2).

3.41 Theorem: A subset A of (X,τ_1,τ_2) is (i,j) pgprw-open iff F$\subseteq$ τ_j-p-int(A),whenever F is

τ_i - rgα-open set and F$\subseteq$ A.

Proof: Suppose that F$\subseteq$ τ_j-p-int(A),whenever F$\subseteq$ A and F is τ_i -rgα-open set and to prove that A is (i,j) pgprw-open.Let G be τ_i - rgα-open and A^c $\subseteq$ G. Then G^c $\subseteq$ A and G^c is τ_i -rgα-open.

if A is rgα-open in(X, τ)then X-A is also rgα-open, then $G^c$$\subseteq$ τ_j-p-int(A) that is

$(\tau_j\text{-p-int}(A))^c \subseteq G$, since τ_j-pcl (A^c) = $[\tau_j\text{-p-int}(A)]^C$ thus A^c is (i,j) pgprw-closed that is A is (i,j) pgprw-open.

Convesely, Suppose that A is (i,j)-pgprw-open, F $\subseteq$ A and F is τ_i - rgα-open. Then $A^c \subseteq F^c$

And F^c is also τ_i - rgα-open. if A is rgα-open in (X, τ), then X-A is also rgα-open.

Since A^c is (i,j)-pgprw-closed, we have (τ_j-p-cl(A^c) $\subseteq$ F^c and so F$\subseteq$ τ_j-p-int(A),since

(τ_j-p-cl(A^c) = $(\tau_j\text{-p-int}(A))^c$.

3.42 Theorem : Let A and G be two subsets of a bitopological spaces(X,τ_1,τ_2).

If the set A is (i,j) pgprw-open, then G=X, whenever G is τ_i - rgα-open and (τ_j-p-int(A) ∪ $A^c$$\subseteq$ G.

Proof: Let A be (i,j) pgprw-open, G be the τ_i - rgα-open and (τ_j-p-int(A)) ∪ $A^c$$\subseteq$ G. Then

$G^C$$\subseteq$ $(\tau_j\text{-p-int}(A) \cup A^c)^c$ = $(\tau_j\text{-p-int}(A)^C$ - A^C. Since A^C is (i,j)pgprw-closed and G^c is τ_i - rgα-open. .w.k.t if A is (i,j) pgprw-closed, then τ_j-pcl (A)- A contains no non-empty τ_i - rgα-open set. It followes that G^C = ∅, therefore G=X.

The converse of the theorem need not be true as seen from the following example.

3.43 Example : Let X=Y={a,b,c}, τ_1 = { X,∅,{a},{b},{a,b}} τ_2 = { X,∅,{a},{b,c}}

If A= {a,b} then the only τ_1 rgα-open set containing τ_2 – p-int(A) ∪ A^c is X. But A is not (1,2)
Pgprw-open set in (X,τ_1,τ_2).

3.44 Theorem: If a subset A of (X, τ_1,τ_2) is (i,j) pgprw-closed, then τ_j-pcl (A) - A is (i,j) pgprw-open.

Proof: Let A be a (i,j) pgprw-closed subset in (X,τ_1,τ_2). Let F be a τ_1 rgα-open set such that
F⊆ τ_j-pcl (A)-A. Since w.k.t If A is (i,j) pgprw-closed then τ_j-pcl (A)- A contains non-empty
τ_1 rgα-open set then τ_j-pcl (A)- A is (i,j) pgprw-open then F = ∅.
Therefore F⊆ τ_j-p-int(τ_j-p-cl(A)-A) and by theorem 3.41 ; τ_j-pcl (A)- A is (i,j) pgprw-open.
The converse of the above theorem need not be true as seen from the following example.

3.45 Example
Let X=Y={a,b,c,d}, τ_1={X,∅,{a},{c,d},{a,c,d}}and
τ_2 = { Y,∅,{a},{b},{a,b},{a,b,c}}for the subset A = {a,b,d} in X. τ_2-pcl (A)- A = X – {a,b,d}= {c}
is (1,2) pgprw-open but A = {a,b,d} is not (1,2) pgprw-closed.

3.46 Theorem: If τ_j-p-int(A) ⊆B⊆A and A is (i,j) pgprw-open in (X, τ_1,τ_2) then B is (i,j) pgprw-open.

Proof: Let F be τ_1 rgα-open s.t F⊆B. Now F⊆B⊆A that is F⊆A. Since F is (i,j) pgprw-open by theorem 3.41 F⊆ τ_j-p-int(A) by hypothesis τ_j-p-int(A) ⊆B.
Therefore τ_j-int(τ_j-p-int(A) ⊆ τ_j-p-int(B). That is τ_j-p-int(A) ⊆τ_j-p-int(B) and
hence F⊆ τ_j-p-int(A) again by thm 3.41;B is a (i,j) pgprw-open set in (X, τ_1,τ_2).

3.47Corollary : Let A and B be subsets of a space (X, τ_1,τ_2) if B is (i,j) pgprw-open and
τ_j-p-int(B)⊆A; then A∩B is (i,j)-pgprw-open.

Proof: Let B be (i,j)pgprw-open and τ_j-p-int(B)⊆A that is τ_j-p-int(B)⊆A then τ_j-p-int(B)⊆ A∩B.
Also τ_j-p-int(B)⊆A∩B⊆B and B is (i,j) pgprw-open by thm 3.46 then A∩B is (i,j)-pgprw-open.

3.48 Theorem: Every singleton point set in a space (X, τ_1,τ_2) is either (i,j) pgprw-open .

Proof: Let (X, τ_1,τ_2) be a bitopological space. Let xϵX to prove {x} is either (i,j) pgprw-open
That is to prove X-{x} is either (i,j) pgprw-closed which follow from the statement i.e, i,j be fixed integers of (1,2) for each x of (X, τ_1,τ_2), {x} is a rgα-open in (X, τ).

4.0 (τ_i, τ_j) pgprw-closure in bitopological spaces

4.1 Defintion: Let (X, τ_1,τ_2) be a bitopological space and i,jϵ{1,2} be fixed integers. For each subset E of X,define (τ_i,τ_j)-pgprw cl(E) = ∩ {A:EϵD_{pgprw} (i,j)-pgprw-cl(E)).

4.2 Theorem: If A and B be subsets of (X, τ_1,τ_2) then
(i) (i,j) pgprw-cl(X)=X and (i,j)-pgprw-cl(∅).
(ii) A ⊆ (i,j)-pgprw-cl(A)
(iii) if B is any (i,j) pgprw-closed set containing A Then (i,j)-pgprw-cl(A) ⊆ B.

Proof: Follows from the defintion 4.1

4.3 Theorem : Let A and B be subsets of (X, τ_1,τ_2) and i,jϵ{1,2} be fixed integers. If A⊆B, then (i,j)-pgprw-cl(A) ⊆ (i,j) pgprw-cl(A) .

Proof: Let A⊆B, by definition 4.1 (i,j)pgprw-cl(B) = ∩ {{F:B⊆FϵD_{pgprw} (i,j).
If B⊆ FϵD_{pgprw},(i,j)since A⊆B, A⊆B⊆FϵD_{pgprw}(i,j),we have (i,j) pgprw-cl(A)⊆F.
Therefore (i,j)-pgprw-cl(A) ⊆∩{F:B⊆FϵD_{pgprw} (i,j)}= (i,j)-pgprw-cl(B).
That is (i,j)-pgprw-cl(A) ⊆ (i,j) pgprw-cl(B).

4.4 Theorem: Let A be a subset of (X, τ_1,τ_2). If τ_1⊆ τ_2 and rgα-open (X, τ_1) ⊆ rgα-open
(X, τ_2),then (1,2) pgprw-cl(A) ⊆ (2,1) pgprw-cl(A).

Proof: By defintion 4.1 (1,2)pgprw-cl(A) = ∩ {{F:A⊆FϵD_{pgprw} (1,2).Since τ_1⊆ τ_2. Since w.k.t, if τ_1⊆ τ_2 and rgα-open (X, τ_1) ⊆ rgα-open (X, τ_2) in (X, τ_1,τ_2) then D_{pgprw} (τ_2,τ_1) ⊆ D_{pgprw} (τ_1,τ_2) this implies D_{pgprw} (2,1) ⊆ D_{pgprw} (1,2) Therefore
∩ {{F:A⊆FϵD_{pgprw} (1,2) ⊆ ∩ {{F:A⊆FϵD_{pgprw} (2,1); that is (1,2)pgprw-cl(A)⊆ ∩{{F:A⊆FϵD_{pgprw} (2,1)= (2,1)-pgprw-cl(A),hence (1,2)pgprw-cl(A) ⊆ pgprw-cl(A).

4.5 Theorem: Let A be a subset of (X, τ_1,τ_2) and i,jϵ{1,2} be fixed integers, then
A⊆(i,j)pgprw-cl(A) ⊆ τ_j-p-cl(A).

Proof: By definition 4.1, it follows that A⊆(i,j)pgprw-cl(A) . Now to prove that
(i,j)-pgprw-cl(A) ⊆ τ_j-p-cl(A) by definition of closure, τ_j-p-cl(A) = {F⊆X:A⊆F and F is τ_j-closed}
If A⊆F and F is τ_j-closed set then F is (i,j)-pgprw-closed,as every τ_j-closed set is (i,j)pgprw-closed.
Therefore (i,j)-pgprw-cl(A)⊆ ∩{F⊆X:A⊆F and F is τ_j-closed} τ_j-p-cl(A) hence (i,j)-pgprw-cl(A) ⊆ τ_j-p-cl(A).

4.6 Remark: Containment relation in the above theorem may be proper as seen from the followin example.

4.7 Example : Let X={a,b,c,d}, τ_1 = { X,∅,{a},{b},{a,b},{a,b,c}} τ_2 = { X,∅,{a,b},{c,d}} then τ_2–closed sets are { X,∅,{a,b},{c,d} and (1,2) pgprw-closed sets are
{X,∅,{c},{d},{a,d},{b,d},{c,d},{a,c,d},{b,c,d},{a,b,d}. Take A= {b,c} then τ_2 –cl(A)=X.
And (1,2) pgprw-cl(A) = {b,c,d}. Now A ⊆ (1,2) pgprw-cl(A),but A ≠ (1,2) pgprw-cl(A)
Also (1,2) pgprw-cl(A) ⊆ τ_2 – cl(A) but (i,j) pgprw-cl(A) ≠ τ_j – cl(A).

4.8 Theorem : Let A be a subset of (X, τ_1,τ_2) and i,jϵ{1,2}be fixed integers. If A is (i,j)pgprw-closed then (i,j)-pgprw-cl(A)=A.

Proof: Let A be a (i,j) pgprw-closed subset of (X, τ_1, τ_2) w.k.t A $\subseteq$(i,j)-pgprw-cl(A) also
A $\subseteq$A and A is (i,j) pgprw-closed by theorem 4.2(iii) (i,j) pgprw-cl(A) $\subseteq$A.
Hence (i,j) pgprw-cl(A)=A.

4.9 Remark: The converse of the above theorem 4.8 need not be true as seen from the following example

4.10 Example: Let X={a,b,c}, τ_1 = { X,∅,{a},{b},{a,b}} τ_2 = { X,∅,{a},{b,c}}then (1,2)
Pgprw-closed sets {b,c} Take A={a} Now (1,2)pgprw-cl(A) = X ∩{a,c}∩{a}={a}
But {a} is not a (1,2) pgprw-closed set.

4.11 Theorem: The operator (i,j)-pgprw-closure in defintion 4.1 (i) is the kurutowski closure opeator on X.

Proof:

(i) Let (i,j) pgprw-cl(∅)=∅ by theorem 4.2(i)
(ii) E$\subseteq$ (i,j)pgprw-cl(E) for any subset E of X by theorem 4.2(ii)
(iii) Suppose E and F are two subsets of (X, τ_1, τ_2)it follows from theorem 4.3 that

(i,j) pgprw-cl(E) $\subseteq$(i,j)-pgprw-cl(E∪F) and that (i,j) pgprw-cl(F)$\subseteq$ (i,j)-pgprw-cl(E∪F);hence
We have (i,j)-pgprw-cl(E) ∪ (i,j) pgprw-cl(F) $\subseteq$ (i,j) pgprw-cl(E∪F).
Now if x ∉ (i,j)-pgprw-cl(E) ∪ (i,j)-pgprw-cl(F) then x ∉(i,j)-pgprw-cl(F),it follows that there exist A,BϵD_{pgprw} (i,j) s.t E$\subseteq$ A. x ∉ A and F$\subseteq$ B, x ∉ B. Hence E∪ F$\subseteq$ A∪B, x ∉ A∪B. Since A∪B is (i,j)Pgprw-closed since w.k.t if A,BϵD_{pgprw} (i,j),then A∪BϵD_{pgprw} (i,j) so x ∉(i,j)-pgprw-cl(E∪F). Then we have (i,j)pgprw-cl(E∪ F) $\subseteq$(i,j) pgprw-cl(E) ∪ (i,j) pgprw-cl(F).
From the above discussions we have (i,j) pgprw-cl(E∪F)= (i,j)-pgprw-cl(E) ∪ (i,j)-pgprw-cl(F).
(iv) Let E be any subset of (X, τ_1, τ_2) by the definition of (i,j)-pgprw-closure,
(i,j)-pgprw-cl(E)=∩{A$\subseteq$X:E$\subseteq$Aϵ D_{pgprw} (i,j)}.If E$\subseteq$Aϵ D_{pgprw} (i,j)} then (i,j) pgprw-cl(E)$\subseteq$A.Since A is a (i,j)-pgprw-closed set containing (i,j)pgprw-cl(E) by theorem 4.2(iii)
(i,j) –pgprw-cl{i,j}-pgprw-cl(E)$\subseteq$A.Hence (i,j) –pgprw-cl{i,j}-pgprw-cl(E) $\subseteq${ A$\subseteq$X: E$\subseteq$Aϵ D_{pgprw} (i,j)}= (i,j)-pgprw-cl(E).
Conversely, (i,j) pgprw-cl(E) $\subseteq$ (i,j) pgprw-cl(i,j)-pgprw-cl(E) is true by theorem 4.2(iii), then we have (i,j) pgprw-cl(E)= (i,j)-pgprw-cl(i,j)-pgprw-cl(E).Hence (i,j)-pgprw-closure is a kuraowski closure operator on X.
From the this theorem (i,j)pgprw-closure defines the new topology on X.

4.12 Definition: Let i,jϵ{1,2} be two fixed integers. Let τ_{pgprw}-(i,j) be topology on X generated by (i,j) pgprw-closure in the usual manner .That is τ_{pgprw}-(i,j) = { E$\subseteq$X: (i,j)-pgprw-cl(E^c)= E^C.

4.13 Theorem: Let (X, τ_1, τ_2) be a bitopological space and {i,j}ϵ{1,2} be two fixed integers, then
$\tau_j \subseteq \tau_{pgprw(i,j)}$.

Proof: Let Gϵ τ_j, It follows that G^c is τ_j-closed by theorem 3.7 G^c is (i,j) pgprw-closed. Therefore (i,j)-pgprw-cl(G^c)= G^c by theorem 4.8 that is Gϵ $\tau_{pgprw(i,j)}$ and hence $\tau_j \subseteq \tau_{pgprw(i,j)}$.

4.14 Remark: Containment relation in the above theorem 4.13 may be proper as seen from the following example.

4.15 Example: Let X={a,b,c}, τ_1 = { X,∅,{a},{b},{a,b}} τ_2 = { X,∅,{a}}then (1,2) pgprw-closed sets are { X,∅,{c},{b,c}}and τ_{pgprw} (1,2) ={ X,∅,{a},{a,b}}.clearly $\tau_2 \subseteq \tau_{pgprw}$ (1,2) but
$\tau_2 \neq \tau_{pgprw}$ (1,2).

4.16 Theorem: Let (X, τ_1, τ_2) be a bitopological space and i,j ϵ{1,2} be two fixed integers. If a subset E of x is (i,j) pgprw-closed then E is τ_{pgprw} closed.

Proof: Let a subset E of X be (i,j) pgprw-closed by theorem 4.8 (i,j) pgprw-cl(E)=E that is
(i,j)-pgprw-cl{E^C}^{C}}= {E^c }^{c} .It follows that $E^c\epsilon$ τ_{pgprw} (1,2). Therefore E is τ_{pgprw} (i,j)-closed.

4.17 Theorem : For any point x of (X, τ_1, τ_2),{x} is τ_i – rgα open or τ_{pgprw} (i,j)-open.

Proof: Let x be any point of (X, τ_1, τ_2),since w.k.t i,j be fixed integers of {1,2} for each x of
(X, τ_1, τ_2),{x} is a – rgα open in (X, τ_i)or {x}^{c} is τ_{pgprw} (i,j)-closed then {x} is τ_i – rgα open
that is {x}^{c} is τ_{pgprw} (i,j)-closed, by above thm 4.16 Therefore {x} is τ_i – rgα open
or τ_{pgprw} (i,j)-open.

4.18 Thorem: If $\tau_1 \subseteq \tau_2$ and rgα open$(X, \tau_1) \subseteq$ rgα open(X, τ_2) in(X, τ_1, τ_2) then
τ_{pgprw} (2,1) $\subseteq$ τ_{pgprw} (1,2).

Proof: let Gϵ τ_{pgprw} (2,1), then (2,1)-pgprw-cl(G^c)=G^c to prove that G τ_{pgprw} (1,2),that is to prove
(1,2)pgprw-cl(G^c)=G^c. Now (1,2)-pgprw-cl(G^c)= ∩ {F$\subseteq$X: $G^C$$\subseteq$F$\epsilon$ D_{pgprw} (1,2) .Since $\tau_1 \subseteq \tau_2$
and rgα open$(X, \tau_1) \subseteq$ rgα open(X, τ_2) by thm 3.22 D_{pgprw} (2,1) $\subseteq$ D_{pgprw} (1,2).
Thus ∩ {F$\subseteq$X: $G^C$$\subseteq$F$\epsilon$ D_{pgprw} (1,2) $\subseteq$ ∩ {F$\subseteq$X: $G^C$$\subseteq$F$\epsilon$ D_{pgprw} (2,1).that is (1,2) pgprw-cl(G^C) $\subseteq$ (2,1)
Pgprw-cl(G^c) and so,(1,2) pgprw-cl(G^C)=G^c.Conversely, G^c $\subseteq$ (1,2)pgprw-cl(G^c) is true by the theorem 4.2(ii) then we have (1,2) pgprw-cl(G^c)= G^c that is Gϵ τ_{pgprw} (1,2) and hence τ_{pgprw} (2,1) $\subseteq$ τ_{pgprw} (1,2).

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