ISSN:0075-5166
Karnatak University Journal of Science
journal homepage: www.kud.ac.in

On the Wiener index of total graph and its complement

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ARTICLE INFO

Article history:
Received date 16 December 2015
Received in revised form 6 January 2016
Accepted date 25 January 2016

Keywords:
Wiener index; line graph; total graph; complement; diameter; distance; degree.

ABSTRACT

Given a simple connected graph G, the *Wiener index* $W(G)$ of G is defined as half the sum of distances over all pairs of vertices of G. In practice, G corresponds to what is known as the *molecular graph* of an organic compound. In this paper, we obtain the Wiener index of total graph and its complement for some standard class of graphs, we give bounds for Wiener index of total graph and its complement also establish Nordhaus-Gaddum type of inequality for this.

1. Introduction

In this paper, we are concerned with finite, undirected, connected, simple graph G with vertex set $V(G)=\{v_1,v_2,...,v_n\}$ and edge set $E(G)=\{v'_1,v'_2,...,v'_m\}$. The distance between two vertices v_i and v_j, denoted by $d(v_i,v_j)$ is the length of the shortest path between the vertices v_i and v_j in G. The shortest $v_i - v_j$ path is often called *geodesic*. The *diameter* $diam(G)$ of a connected graph G is the length of any longest geodesic. The *degree* [7]of a vertex v_i in G is the number of edges incident to v_i and is denoted by $d_i = deg(v_i)$.

The *Wiener index* is a graph invariant that belongs to the molecules structure-descriptors called topological indices, which are used for the design of molecules with desired properties.

The *Wiener index*(or *Wiener number*)[11] of a graph G, denoted by $W(G)$ is the sum of distances between all (unordered) pairs of vertices of G, that is

$$W(G)=\sum_{i<j} d(v_i\ v_j)$$

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The *Wiener index* $W(G)$ of the graph G [3,6] is also expressedas

$$W(G)=\frac{1}{2}\sum_{v_i,v_j \in V(G)} d(v_i\ v_j)$$

where the summation is over all possible pairs $v_i, v_j \in V(G)$.

Line graphs, total graphs and middle graphs are widely studied transformation graphs. Let $G=(V(G),E(G))$ be a graph. The *line graph* $L(G)$[1] of G is the graph whose vertex set is $E(G)$ in which two vertices are adjacent if and only if they are adjacent in G. The vertices and edges of a graph are called its *elements*. Two elements of a graph are *neighbors* if they are either incident or adjacent. The *total graph* $T(G)$ [7] has vertex set $V(G)\cup E(G)$, and two vertices of $T(G)$ are adjacent whenever they are neighbors in G. If G is a (n,m) graph whose vertices have degrees d_i, then the total graph $T(G)$ has $n_T = n+m$ vertices and $m_T = 2m+\frac{1}{2}\sum_{i=1}^{n} d_i^2$ edges. The *complement* of G, denoted by $\overline{G}$, is the graph with the same vertex set as G, where two vertices are adjacent if and only if they are not adjacent in G. We denote the *complement of total graph* $T(G)$ of G by $\overline{T}(G)$. The complement of total graph $\overline{T}(G)$ is a graph whose vertex set is $V(G)\cup E(G)$, and two vertices of $\overline{T}(G)$ are adjacent whenever they are not neighbors in G. A tree is called a double star[2] $S_{p,q}$ if it obtained from S_p and S_q by connecting the center of S_p with that of S_q via an edge. For notations and undefined terminologies we follow [7].

The following Theorems are useful for proving our main results.

Theorem 1.1 [8]. *For every tree T of order n,*

$$W(L(T)) = W(T) - \binom{n}{2}.$$

Theorem 1.2 [4]. *Let G be a connected graph with minimum degree $\delta(G)\ge 2$. Then $W(G)\le W(L(G))$. Equality holds only for cycles.*

Theorem 1.3 [9]. *If G is a (n,m) graph with $diam(G)\le 2$, then*

$$W(G)=n(n-1)-m.$$

Theorem 1.4 [10]. *If P_n is a path of order n, then*

$$W(P_n)=\frac{n^3-n}{6}.$$

Theorem 1.5 [10]. *If S_n is a star of order n, then* $W(S_n)=(n-1)^2$.

Theorem 1.6 [10]. *If C_n is a cycle of order n, then*

$$W(C_n)=\begin{cases}\dfrac{n^3}{8} & \text{if } n \text{ is even.}\\ \dfrac{n^3-n}{8} & \text{if } n \text{ is odd.}\end{cases}$$

Theorem 1.7 [2]. *For any tree T of order n,* $W(S_n)\le W(T)\le W(P_n)$.

Theorem 1.8 [5]. *Let G be a connected graph with n vertices and m edges. Then*

$$\sum_{i=1}^{n} \mathrm{d}_i^2 = m\left[\frac{2m}{n-1}+(n-2)\right]$$ if and only if G is a star graph or a complete graph.

Theorem 1.9 [9] *For any graph G of order n, size m with $diam(G)\ge 3$,*

$$W(G)\ge n^2-n-m+1$$

holds. Further, the equality holds, if G contains exactly two vertices of eccentricity three and rest are of eccentricity two.

2. Results

Theorem 2.1 *Let T be a tree of order n. Then*

$$W(T(T)) = 4W(T) - \binom{n}{2}.$$

Proof. Let $V=\{v_1,v_2,...,v_n\}$ be the vertex set and $E=\{v'_1,v'_2,...,v'_{n-1}\}$ be the edge set of a tree T. Then $T(T)$ is the total graph of tree T with the vertex set $V'=V\cup\{v'_1,v'_2,...,v'_{n-1}\}$, where v'_i is vertex of a total graph corresponding to the edge of tree. Splitting the summation of Wiener index of $T(T)$ into four parts,

$W(T(T))$=half of the shortest distance between the vertices v_i and v_j

+half of the shortest distance between the vertices v'_i and v'_j

+half of the shortest distance between the vertices v_i and v'_j

+half of the shortest distance between the vertices v'_i and v_j

$$W\left(T(T)\right)=\frac{1}{2}\sum_{v_i,v_j\in V'}d\left(v_i,v_j\right)+\frac{1}{2}\sum_{v'_i,v'_j\in V'}d\left(v'_i,v'_j\right)$$

$$+\frac{1}{2}\left\{\begin{array}{l} d\left(v_1,v'_1\right)+d\left(v_1,v'_2\right)+\ldots+d\left(v_1,v'_{n-1}\right)\\ +d\left(v_2,v'_1\right)+d\left(v_2,v'_2\right)+\ldots+d\left(v_2,v'_{n-1}\right)\\ +\ldots\ldots\ldots+\ldots\ldots\ldots\ldots+\ldots\ldots\ldots\ldots+\\ +d\left(v_n,v'_1\right)+d\left(v_n,v'_2\right)+\ldots+d\left(v_n,v'_{n-1}\right)\end{array}\right\}$$

$$+\frac{1}{2}\left\{\begin{array}{l} d\left(v'_1,v_1\right)+d\left(v'_1,v_2\right)+\ldots+d\left(v'_1,v_n\right)\\ +d\left(v'_2,v_1\right)+d\left(v'_2,v_2\right)+\ldots+d\left(v'_2,v_n\right)\\ +\ldots\ldots\ldots+\ldots\ldots\ldots\ldots+\ldots\ldots\ldots\ldots+\\ +d\left(v'_{n-1},v_1\right)+d\left(v'_{n-1},v_2\right)+\ldots+d\left(v'_{n-1},v_n\right)\end{array}\right\}$$

$$W\left(T(T)\right)=\frac{1}{2}\sum_{v_i,v_j\varepsilon V},d\left(v_i\ v_j\right)+\frac{1}{2}\sum_{v'_i,v'_j\varepsilon E}d\left(v'_i,v'_j\right)$$

$$+\frac{1}{2}\left\{\begin{array}{l} d\left(v_1,v_1\right)+d\left(v_1,v_2\right)+\ldots+d\left(v_1,v_{n-1}\right)\\ +d\left(v_2,v_1\right)+d\left(v_2,v_2\right)+\ldots+d\left(v_2,v_{n-1}\right)\\ +\ldots\ldots\ldots+\ldots\ldots\ldots\ldots+\ldots\ldots\ldots\ldots+\\ +d\left(v_n,v_1\right)+d\left(v_n,v_2\right)+\ldots+d\left(v_n,v_{n-1}\right)\end{array}\right\}$$

$$+\frac{1}{2}\left\{\begin{array}{l} d\left(v_1,v_1\right)+d\left(v_1,v_2\right)+\ldots+d\left(v_1,v_n\right)\\ +d\left(v_2,v_1\right)+d\left(v_2,v_2\right)+\ldots+d\left(v_2,v_n\right)\\ +\ldots\ldots\ldots+\ldots\ldots\ldots\ldots+\ldots\ldots\ldots\ldots+\\ +d\left(v_{n-1},v_1\right)+d\left(v_{n-1},v_2\right)+\ldots+d\left(v_{n-1},v_n\right)\end{array}\right\}$$

$$W\left(T(T)\right)=\frac{1}{2}\sum_{v_i,v_j\in V}d\left(v_i,v_j\right)+\frac{1}{2}\sum_{v'_i,v'_j\in E}d\left(v'_i,v'_j\right)$$

$$+\frac{1}{2}\sum_{v_i,v_j\in V}\mathrm{d}\left(v_i,v_j\right)+\frac{1}{2}\sum_{v_i,v_j\in V}d\left(v_i,v_j\right)$$

$$W\left(T(T)\right)=W(T)+W\left(L(T)\right)+W(T)+W(T).$$

From Theorem 1.1, we have

$$W\left(T(T)\right)=4W(T)-\binom{n}{2}.$$

Corollary 2.2 *Let S_n be a star graph of order n.*

Then $W\left(T(S_n)\right)=\dfrac{7n^2-15n+8}{2}$

Proof. A star S_n is also a tree hence from Theorem 2.1, we have

$$W\left(T(S_n)\right)=4W(S_n)-\binom{n}{2}.$$

From Theorem 1.5,

$$W\left(T(S_n)\right)=4(n-1)^2-\frac{n(n-1)}{2}=\frac{7n^2-15n+8}{2}.$$

Corollary 2.3 *Let P_n be a path of order n. Then* $W\left(T(P_n)\right)=\dfrac{n\left(4n^2-3n-1\right)}{6}$.

Proof. A path P_n is also a tree hence from Theorem 2.1, we have

$$W\left(T(P_n)\right)=4W(P_n)-\binom{n}{2}.$$

From Theorem 1.4, we have

$$W\left(T(P_n)\right)=4\left(\frac{n^3-n}{6}\right)=\frac{n\left(4n^2-3n-1\right)}{6}.$$

Corollary 2.4 *For any tree T of order n,*

$$W\left(T(S_n)\right)\le W\left(T(T)\right)\le W\left(T(P_n)\right).$$

Proof. From Theorem 1.7, we have

$$W(S_n)\le W(T)\le W(P_n)$$

$$4W(S_n)-\binom{n}{2}\le 4W(T)-\binom{n}{2}\le 4W(P_n)-\binom{n}{2}.$$

From Theorem 2.1, Corollaries 2.2 and 2.3, we have

$W(T(S_n)) \leq W(T(T)) \leq W(T(P_n))$.

Theorem 2.5 *Let* C_n *be the cycle of ordern. Then* $W(T(C_n)) = \frac{n^2(n+1)}{2}$.

Proof. Let $V = \{v_1, v_2, ..., v_n\}$ be the vertex set and $E = \{v'_1, v'_2, ..., v'_n\}$ be the edge set of C_n. Then $T(C_n)$ is the total graph of cycle C_n with vertex set $V' = V \cup \{v'_1, v'_2, ..., v'_n\}$, where v'_i is vertex of a total graph corresponding to the edge of cycle. Splitting the summation of Wiener index of $T(C_n)$ into four parts,

$W(T(C_n))$ = half of the shortest distance between the vertices v_i and v_j

+ half of the shortest distance between the vertices v'_i and v'_j

+ half of the shortest distance between the vertices v_i and v'_j

+ half of the shortest distance between the vertices v'_i and v_j

$$W(T(C_n)) = \frac{1}{2}\sum_{v_i, v_j \in V'} d(v_i, v_j) + \frac{1}{2}\sum_{v'_i, v'_j \in V'} d(v'_i, v'_j)$$

$$+\frac{1}{2}\left\{\begin{array}{l} d(v_1, v'_1) + d(v_1, v'_2) + ... + d(v_1, v'_n) \\ +d(v_2, v'_1) + d(v_2, v'_2) + ... + d(v_2, v'_n) \\ +..........+...........+...........+.............+ \\ +d(v_n, v'_1) + d(v_n, v'_2) + ... + d(v_n, v'_n) \end{array}\right\}$$

$$+\frac{1}{2}\left\{\begin{array}{l} d(v'_1, v_1) + d(v'_1, v_2) + ... + d(v'_1, v_n) \\ +d(v'_2, v_1) + d(v'_2, v_2) + ... + d(v'_2, v_n) \\ +..........+.................+...............+ \\ +d(v'_n, v_1) + d(v'_n, v_2) + ... + d(v'_n, v_n) \end{array}\right\}$$

$$W(T(C_n)) = \frac{1}{2}\sum_{v_i, v_j \in V} d(v_i, v_j) + \frac{1}{2}\sum_{v'_i, v'_j \in E} d(v'_i, v'_j)$$

$$+\frac{1}{2}\left\{\begin{array}{l} d(v_1, v_1) + d(v_1, v_2) + ... + d(v_1, v_n) + diam(T(C_n)) \\ +d(v_2, v_1) + d(v_2, v_2) + ... + d(v_2, v_n) + diam(T(C_n)) \\ +................+......................+......................+ \\ +d(v_n, v_1) + d(v_n, v_2) + ... + d(v_n, v_n) + diam(T(C_n)) \end{array}\right\}$$

$$+\frac{1}{2}\left\{\begin{array}{l} d(v_1, v_1) + d(v_1, v_2) + ... + d(v_1, v_n) + diam(T(C_n)) \\ +d(v_2, v_1) + d(v_2, v_2) + ... + d(v_2, v_n) + diam(T(C_n)) \\ +................+......................+......................+ \\ +d(v_n, v_1) + d(v_n, v_2) + ... + d(v_n, v_n) + diam(T(C_n)) \end{array}\right\}$$

$$W(T(C_n)) = W(C_n) + W(L(C_n)) + 2\left\{\sum_{v_i, v_j \in V} d(v_i\ v_j) + n\ diam(T(C_n))\right\}$$

From Theorem 1.2, we have

$$W(T(C_n)) = W(C_n) + W(C_n) + 2\left\{W(C_n) + \frac{n\ diam(T(C_n))}{2}\right\}$$

$$W(T(C_n)) = 4W(C_n) + n\ diam(T(C_n)).$$

Case 1. For an odd cycle, $diam(T(C_n)) = \frac{n+1}{2}$ and from Theorem 1.6, we have

$$W(T(C_n)) = 4\left(\frac{n^3 - n}{8}\right) + n\left(\frac{n+1}{2}\right) = \frac{n^2(n+1)}{2}.$$

Case 2. For an even cycle, $diam(T(C_n)) = \frac{n}{2}$ and from Theorem 1.6, we have

$$W(T(C_n)) = 4\left(\frac{n^3}{8}\right) + \frac{n^2}{2} = \frac{n^2(n+1)}{2}.$$

From the above two cases, we have

$$W(T(C_n)) = \frac{n^2(n+1)}{2}.$$

Theorem 2.6 *If K_n is a complete graph of order n, then* $W\left(T\left(K_n\right)\right)=\dfrac{n^2\left(n^2-1\right)}{4}$.

Proof. Let K_n be a complete graph with *n* vertices and $m=\dfrac{n(n-1)}{2}$ edges. Then from Theorem 1.8, $\sum_{i=1}^{n} d_i^2 = n(n-1)^2$ and $T(K_n)$ has $n_T=\dfrac{n^2+n}{2}$ vertices and $m_T=\dfrac{n^3-n}{2}$ edges.

In $T(K_n)$ distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam(T(K_n))=2$.

From Theorem 1.3, we obtain the result.

Next, we determine the Wiener index of total graph of wheel. The wheel[7] invented by W. T. Tutte. For $n \geq 4$, wheel W_n is defined to be the graph K_1+C_{n-1}.

Theorem 2.7 *If W_n is a wheel graph of order $n \geq 4$, then*

$$W\left(T\left(W_n\right)\right)=\frac{17n^2-45n+28}{2}.$$

Proof. Let W_n be a wheel graph with $n \geq 4$ vertices, $m=2(n-1)$ edges and one vertex is of degree $(n-1)$ and remaining $(n-1)$ vertices are of degree 3. Then $\sum_{i=1}^{n} d_i^2=(n-1)(n+8)$ and $T(W_n)$ has $n_T=3n-2$ vertices and $m_T=\dfrac{n^2+15n-16}{2}$ edges.

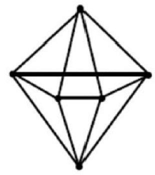
In $T(W_n)$, distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam(T(W_n))=2$.

From Theorem 1.3, $W\left(T\left(W_n\right)\right)=n_T\left(n_T-1\right)-m_T$

$$W\left(T\left(W_n\right)\right)=(3n-2)(3n-3)\cdot -\left(\frac{n^2+15n-16}{2}\right)=\frac{17n^2-45n+28}{2}$$

Next, we determine the Wiener index of total graph of bipyramid graph. For $n \geq 5$, bipyramid graph [6] is defined to be the graph $C_{n-2}+\left(P_1 \cup P_1\right)$.

$C_4+(P_1 \cup P_1)$:



Theorem 2.8 *If G is a bipyramid graph of order $n \geq 5$, then*

$$W\left(T\left(G\right)\right)=15n^2-62n+66.$$

Proof. Let G be a bipyramid graph with $n \geq 5$ vertices and $m=3(n-2)$ edges, and two vertices are of degree $(n-2)$ and remaining $(n-2)$ vertices are of degree 4. Then $\sum_{i=1}^{n} d_i^2=2(n-2)(n+6)$ and $T(G)$ has $n_T=4n-6$ vertices and $m_T=n^2+10n-24$ edges.

In total graph of bipyramid graph distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam\left(T\left(G\right)\right)=2$.

From Theorem 1.3, we have

$$W\left(T\left(G\right)\right)=n_T\left(n_T-1\right)-m_T$$

$$W\left(T\left(G\right)\right)=(4n-6)(4n-7)-\left[n^2+10n-24\right]=15n^2-62n+66$$

Theorem 2.9 *Let $T^* \neq S_n, S_{p,q}$ be a tree of order n. Then*

$$W\left(\overline{T}\left(T^*\right)\right)=2n^2-n-1+\frac{1}{2}\sum_{i=1}^{n} d_i^2$$

Proof. In $\overline{T}\left(T^*\right)$ the distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam\left(\overline{T}\left(T^*\right)\right)=2$. From Theorem 1.3, we have

$W\left(\overline{T}\left(T^*\right)\right)=n_{\overline{T}}\left(n_{\overline{T}}-1\right)-m_{\overline{T}}$.

Here $n_{\overline{T}}=n+m$ and

$$m_{\overline{T}}=\binom{n+m}{2}-\left(2m+\frac{1}{2}\sum_{i=1}^{n}d_i^2\right)$$

Therefore $W\left(\overline{T}\left(T^*\right)\right)=\binom{n+m}{2}+2m+\frac{1}{2}\sum_{i=1}^{n}d_i^2$.

For tree, $m=n-1$, therefore

$$W\left(\overline{T}\left(T^*\right)\right)=\binom{2n-1}{2}+2\left(n-1\right)+\frac{1}{2}\sum_{i=1}^{n}d_i^2$$

$$W\left(\overline{T}\left(T^*\right)\right)=2n^2-n-1+\frac{1}{2}\sum_{i=1}^{n}d_i^2.$$

If T^* is a star, then $\overline{T}\left(T^*\right)$ has one isolated vertex. Hence $\overline{T}\left(T^*\right)$ is disconnected, obviously Wiener index is not defined.

Corollary 2.10 *If P_n is a path of order $n\geq 5$, then* $W\left(\overline{T}\left(P_n\right)\right)=2n^2+n-4$.

Proof. For $n\geq 5$ path is not a double star. From Theorem 2.9, we have

$$W\left(\overline{T}\left(P_n\right)\right)=2n^2-n-1+\frac{1}{2}\sum_{i=1}^{n}d_i^2$$

For path, $m=n-1$, two vertices of degree one and remaining $(n-2)$ vertices of degree two. Therefore $\sum_{i=1}^{n}d_i^2=2(1)^2+(n-2)2^2=2(2n-3)$.

$$W\left(\overline{T}\left(P_n\right)\right)=2n^2-n-1+\frac{1}{2}2\left(2n-3\right)$$

$$W\left(\overline{T}\left(P_n\right)\right)=2n^2+n-4.$$

Theorem 2.11 *If $S_{p,q}$ is a double star of order n, then*

$$W\left(\overline{T}\left(S_{p,q}\right)\right)=2n^2-n+\frac{1}{2}\sum_{i=1}^{n}d_i^2.$$

Proof. In $\overline{T}\left(S_{p,q}\right)$, distance between two nonpendant vertices is three and all other pair of vertices is two. Therefore $\overline{T}\left(S_{p,q}\right)$ contain exactly two vertices of eccentricity three and rest are of eccentricity two. From Theorem 1.9, we have

$$W\left(\overline{T}\left(S_{p,q}\right)\right)=n_{\overline{T}}^2-n_{\overline{T}}-m_{\overline{T}}+1$$

$$W\left(\overline{T}\left(S_{p,q}\right)\right)=2n^2-n+\frac{1}{2}\sum_{i=1}^{n}d_i^2.$$

Corollary 2.12 *For any nontrivial tree T of order* $n\geq 4$, $W\left(\overline{T}\left(P_n\right)\right)\leq W\left(\overline{T}\left(T\right)\right)$.

Proof. The value of $\sum_{i=1}^{n}d_i^2$ is least for path among all T^*.

Therefore $W\left(\overline{T}\left(P_n\right)\right)\leq W\left(\overline{T}\left(T^*\right)\right)$.

From Theorems 2.9 and 2.11, it is clear that $W\left(\overline{T}\left(T^*\right)\right)\leq W\left(\overline{T}\left(S_{p,q}\right)\right)$.

Therefore $W\left(\overline{T}\left(P_n\right)\right)\leq W\left(\overline{T}\left(T^*\right)\right)\leq W\left(\overline{T}\left(S_{p,q}\right)\right)$.

Obviously, $W\left(\overline{T}\left(P_n\right)\right)\leq W\left(\overline{T}\left(T\right)\right)$.

Theorem 2.13 *If C_n is a cycle of order $n\geq 4$, then* $W\left(\overline{T}\left(C_n\right)\right)=n\left(2n+3\right)$.

Proof. In $\overline{T}\left(C_n\right)$ distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam\left(\overline{T}\left(C_n\right)\right)=2$.

From Theorem 1.3, we have

$$W\left(\overline{T}\left(C_n\right)\right)=n_{\overline{T}}\left(n_{\overline{T}}-1\right)-m_{\overline{T}}.$$

Here $n_{\overline{T}}=n+m$ and

$$m_{\overline{T}}=\binom{n+m}{2}-\left(2m+\frac{1}{2}\sum_{i=1}^{n}d_i^2\right)$$

Therefore $W\left(\overline{T}\left(C_n\right)\right)=\binom{n+m}{2}+2m+\frac{1}{2}\sum_{i=1}^{n}d_i^2$

For cycle $m=n$ and all vertices of degree 2. Therefore

$$\sum_{i=1}^{n}d_i^2=2^2n=4n.$$

Thus $W\left(\overline{T}\left(C_n\right)\right)=\binom{2n}{2}+2n+\frac{4n}{2}=n\left(2n+3\right)$.

Theorem 2.14 *If* K_n *is a complete graph of order* $n \geq 4$, *then* $W\left(\overline{T}\left(K_n\right)\right) = \frac{n(n-1)(n+1)(n+6)}{8}$.

Proof. In $\overline{T}\left(K_n\right)$ the distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam\left(\overline{T}\left(K_n\right)\right) = 2$.

From Theorem1.3, we have

$$W\left(\overline{T}\left(K_n\right)\right) = n_{\overline{T}}\left(n_{\overline{T}} - 1\right) - m_{\overline{T}}.$$

Here $n_{\overline{T}} = n + m$ and

$$m_{\overline{T}} = \binom{n+m}{2} - \left(2m + \frac{1}{2}\sum_{i=1}^{n} d_i^2\right)$$

Therefore $W\left(\overline{T}\left(K_n\right)\right) = \binom{n+m}{2} + 2m + \frac{1}{2}\sum_{i=1}^{n} d_i^2$.

For complete graph $m = \frac{n(n-1)}{2}$ and from Theorem 1.8, we have $\sum_{i=1}^{n} d_i^2 = n(n-1)^2$.

$$W\left(\overline{T}\left(K_n\right)\right) = \frac{n(n-1)(n+1)(n+6)}{8}.$$

Theorem 2.15 *If* W_n *is a wheel of order* $n \geq 4$, *then* $W\left(\overline{T}\left(W_n\right)\right) = 5(n-1)(n+1)$.

Proof. In $\overline{T}\left(W_n\right)$ the distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam\left(\overline{T}\left(W_n\right)\right) = 2$.

From Theorem1.3, we have

$$W\left(\overline{T}\left(W_n\right)\right) = n_{\overline{T}}\left(n_{\overline{T}} - 1\right) - m_{\overline{T}}.$$

Here $n_{\overline{T}} = n + m$ and

$$m_{\overline{T}} = \binom{n+m}{2} - \left(2m + \frac{1}{2}\sum_{i=1}^{n} d_i^2\right)$$

$$W\left(\overline{T}\left(W_n\right)\right) = \binom{n+m}{2} + 2m + \frac{1}{2}\sum_{i=1}^{n} d_i^2$$

For wheel $m = 2(n-1)$, one vertex is of degree $n-1$ and $n-1$ vertices of degree three.

Therefore

$$\sum_{i=1}^{n} d_i^2 = 1(n-1)^2 + (n-1)3^2 = (n-1)(n+8)$$

$$W\left(\overline{T}\left(W_n\right)\right) = \binom{3n-2}{2} + \left[4(n-1) + \frac{1}{2}(n-1)(n+8)\right]$$

$$W\left(\overline{T}\left(W_n\right)\right) = 5(n-1)(n+1).$$

Theorem 2.16 *Let* G *be a bipyramid graph of order* $n \geq 5$. *Then*

$$W\left(\overline{T}(G)\right) = 9n^2 - 16n - 3.$$

Proof. In $\overline{T}(G)$ the distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam\left(\overline{T}(G)\right) = 2$.

From Theorem 1.3, we have

$$\mathrm{W}\left(\overline{T}(G)\right) = n_{\overline{T}}\left(n_{\overline{T}} - 1\right) - m_{\overline{T}}.$$

Here $n_{\overline{T}} = n + m$ and

$$m_{\overline{T}} = \binom{n+m}{2} - \left(2m + \frac{1}{2}\sum_{i=1}^{n} d_i^2\right)$$

$$W\left(\overline{T}\left(W_n\right)\right) = \binom{n+m}{2} + 2m + \frac{1}{2}\sum_{i=1}^{n} d_i^2$$

For bipyramid graph $m = 3(n-2)$ and two vertices of degree $n-2$ and remaining $n-2$ vertices of degree four.

Therefore

$$\sum_{i=1}^{n} d_i^2 = 2(n-2)^2 + (n-2)4^2 = 2(n-2)(n+6).$$

Thus

$$W(\bar{T}(G)) = \binom{4n-6}{2} + 2[3(n-2)] + \frac{1}{2}2(n-2)(n+6)$$

$$W(\bar{T}(G)) = 9n^2 - 16n - 3.$$

Theorem 2.17 *If G is a connected graph of order $n \geq 2$, then $W(G) < W(T(G))$.*

Proof. Let G be a connected graph with n vertices and m edges. Then $T(G)$ is a total graph with $n_T = n+m$ and $m_T = 2m + \frac{1}{2}\sum_{i=1}^{n} d_i^2$ edges.

Wiener index of a graph increases when new vertices are added to it and G is induced subgraph of $T(G)$. Therefore $W(G) < W(T(G))$.

Lemma 2.18 *If G is a connected graph of order n, then*

$$\frac{7n^2 - 15n + 8}{2} \leq W(T(G)) \leq \frac{n^2(n^2-1)}{4}.$$

Upper bound attains if G is complete graph and lower bound attains if G is a star graph.

Proof. Let G be a connected graph with n vertices and m edges. Then $T(G)$ be the total graph with $n_T = n+m$ vertices and $m_T = 2m + \frac{1}{2}\sum_{i=1}^{n} d_i^2$ edges.

For any graph G has maximum number of edges if and only if $G \cong K_n$ and $T(G)$ has maximum number of vertices if and only if $G \cong K_n$.

As Wiener index of a graph increases when new vertices are added to a graph, and $T(K_n)$ has maximum number number of vertices compared to any other $T(G)$.
$W(T(G)) \leq W(T(K_n))$.

From Theorem 2.6, $W(T(G)) \leq \frac{n^2(n^2-1)}{4}$. (1)

Similarly, $W(T(T)) \leq W(T(G))$.

From Corollary 2.4, $W(T(S_n)) \leq W(T(T))$.

It follows that, $W(T(S_n)) \leq W(T(G))$.

From Corollary 2.2, $\frac{7n^2 - 15n + 8}{2} \leq W(T(G))$. (2)

From (1) and (2)we have the result.

Lemma 2.19 *Let G is a connected graph of order $n \geq 5$. Then*

$$2n^2 + n - 4 \leq W(\bar{T}(G)) \leq \frac{n(n-1)(n+1)(n+6)}{8}.$$

Upper bound attains if G is complete graph and lower bound attains if G is a path graph.

Proof. Let G be connected graph with $n \geq 5$ vertices and m edges. Then $\bar{T}(K_n)$ has $n_{\bar{T}} = n+m$ vertices and $m_{\bar{T}} = \binom{n+m}{2} - \left(2m + \frac{1}{2}\sum_{i=1}^{n} d_i^2\right)$ edges.

If G has maximum edges if and only if $G \cong K_n$, then $\bar{T}(G)$ has maximum number of vertices if and only if $G \cong K_n$.

Wiener index of a graph increases when new vertices are added to the graph and $\bar{T}(K_n)$ has maximum number of vertices compared to any other $\bar{T}(G)$.

Therefore $W(\bar{T}(G)) \leq W(\bar{T}(K_n))$.

From Theorem 2.14,

$$W(\bar{T}(K_n)) = \frac{n(n-1)(n+1)(n+6)}{8}.$$

Therefore $W(\bar{T}(G)) \leq \frac{n(n-1)(n+1)(n+6)}{8}$. (3)

Similarly, $W(\bar{T}(T)) \leq W(\bar{T}(G))$

From Corollary 2.12, $W\left(\bar{T}\left(P_n\right)\right) \le W\left(\bar{T}\left(T\right)\right)$.

Obviously, $W\left(\bar{T}\left(P_n\right)\right) \le W\left(\bar{T}\left(G\right)\right)$.

From Corollary 2.10, we have

$$2n^2 + n - 4 \le W\left(\bar{T}\left(G\right)\right). \qquad (4)$$

From (3) and (4),

$$2n^2 + n - 4 \le W\left(\bar{T}\left(G\right)\right) \le \frac{n(n-1)(n+1)(n+6)}{8}.$$

The following theorem gives the Nordhaus-Gaddum [12] type inequality for Wiener index of total graph.

Theorem 2.20 *If* G *is a connected graph of order* $n \ge 5$, *then*

$$\frac{n(11n-13)}{2} \le W\left(T(G)\right) + W\left(\bar{T}(G)\right) \le \frac{3n(n-1)(n+1)(n+2)}{8}$$

Proof. From Lemmas 2.18 and 2.19, we have

$$\frac{7n^2 - 15n + 8}{2} + 2n^2 + n - 4 \le W\left(T(G)\right) + W\left(\bar{T}(G)\right) \le \frac{n^2\left(n^2-1\right)}{4} + \frac{n(n-1)(n+1)(n+6)}{8}$$

$$\frac{n(11n-13)}{2} \le W\left(T(G)\right) + W\left(\bar{T}(G)\right) \le \frac{3n(n-1)(n+1)(n+2)}{8}$$

3. Acknowledgements

This research is supported by UGC-UPE (Non-NET)-Fellowship, K. U. Dharwad, No. KU/Sch/UGC-UPE/ 2014-15/895.This research is also supported by UGC- National Fellowship (NF) New Delhi. No. F./2014-15/NFO-2014-15-OBC-KAR-25873/(SA-III/Website).

4. References

[1] X. An, B. Wu, Hamiltonicity of complements of middle graphs, *Discrete Mathematics*, **307** (2007) 1178-1184.

[2] X. An, B. Wu, The Wiener index of kth power of a graph, *Applied mathematics Letters*. **21** (2008) 436-440.

[3] R . Balakrishan, N. Shridharan, K. Vishwanathanlyer, A sharp lower bound for the Wiener index of a graph, *Manuscript*.

[4] N. Cohen, D. Dimitrov, R. Krakovski, R. Skrekovski, V. Vulkasinovic, On Wiener index of graphs and their line graphs, *Inst. Math. phys. Mech.* **48** (2010) 113.

[5] K. C. Das, Sharp bounds for the sum of the degrees of a graph, *Kragujevac J. Math*. **25** (2003) 31-49.

[6] I. Gutman, Y. N. Yeh, S. L. Lee, Y. L. Luo, Some recent results in the theory of Wiener number, *Indian J. Chem*. **32A** (1993) 651-661.

[7] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass, (1969).

[8] H. S. Ramane, D. S. Revankar, A. B. Ganagi, On the Wiener index of Graph, *J. Indones. Math. Soc.* **18(1)** (2012) 57-66.

[9] H. B. Walikar, V. S. Shigehalli, H. S. Ramane, Bounds on the Wiener index of a graph, *MATCH Commun. Math. Comput. Chem*. **50** (2004) 117-132.

[10] Weisstein Ericw, *Wiener index*, From Math World-A Wolform Web Resource.

[11] H. Wiener, Structural determination of paraffin boiling points, *J. Amer. Chem. Soc*. **69**(1947) 17-20.

[12] L. Zhang, B. Wu, The Nordhaus-Gaddum-type inequalities for some chemical indices, *MATCH Commun. Math. Comput. Chem*. **54** (2005) 189-194.

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