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Neighbourly Irregular Derived Graphs

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ABSTRACT

A connected graph G is said to be neighbourly irregular graph if no two adjacent vertices of G have same degree. In this paper, we obtain neighbourly irregular derived graphs such as semitotal-point graph, *k-th* semitotal-point graph, semitotal-line graph, paraline graph, quasi-total graph and quasivertex-total graph and also neighbourly irregular of some graph products.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we are concerned with finite, simple, connected graph G with vertex set $V(G)=\{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)=\{e_1, e_2, \ldots, e_m\}$. If v_i and v_j are vertices of G, then the edge connecting them will be denoted by v_iv_j. The *degree* of a vertex v in G is denoted by $d_G(v)$. The *complement* of G, denoted by $\overline{G}$, is a graph which has the same vertex set as G, in which two vertices are adjacent if and only if they are not adjacent in G and $d_{\overline{G}}(v) = n - 1 - d_G(v)$ holds for all $v \in V(G)$. Definitions not given here may be found in [4].

A graph G is said to be *regular* if all its vertices have the same degree. A connected graph G is said to be *highly irregular* if each neighbor of any vertex has different degree [1]. The graph G is said to be *neighbourly irregular graph*, abbreviated as *NI* graph, if no

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two adjacent vertices of G have the same degree. This concept was introduced by Bhragsam and Ayyaswamy [2]. In [2, 12], authors constructed *NI* graphs of order n for a given n and a partition of n with distinct parts and proved some properties of *NI* graphs related to graphoidal covering number, gracefulness, ply number, lace number, clique graph, minimal edge covering and studied the neighbourly irregularity of some graph products.

The *line graph* $L(G)$ of a graph G is the graph with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G have a vertex in common. The *subdivision graph* $S(G)$ of a graph G whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and other is an edge of G incident with it.

2. DERIVED GRAPHS

In this paper we considered the following graphs derived from the parent graph G:

1. The ***semitotal-point graph*** $T_2(G)$ as the graph [8] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if (i) they are adjacent vertices of G or (ii) one is a vertex of G and other is an edge of G incident with it. If u is a vertex of G, then $d_{T_2(G)}(u) = 2d_G(u)$. If e is an edge of G, then $d_{T_2(G)}(e) = 2$.
2. The ***k-th semitotal-point graph*** $T_2^k(G)$ of G [6] is the graph obtained by adding k vertices to each edge of G and joining them to the endvertices of the respective edge. Obviously, this is equivalent to adding k triangles to each edge of G.
3. The ***semitotal-line graph*** $T_1(G)$ as the graph [8] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if (i) they are adjacent edges of G or (ii) one is a vertex of G and other is an edge of G incident with it. If u is a vertex of G, then $d_{T_1(G)}(u) = d_G(u)$. If $e=uv$ is an edge of G, then $d_{T_1(G)}(e) = d_G(u) + d_G(v)$.
4. The ***paraline graph*** $PL(G)$ is a line graph of subdivision graph of G.
5. The ***quasi-total graph*** $P(G)$ as the graph [9] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if (i) they are nonadjacent vertices of G or (ii) they are adjacent edges of G or (iii) one is a vertex of G and other is an edge of G incident with it. If u is a vertex of G, then $d_{P(G)}(u) = n - 1$. If $e = uv$ is an edge of G, then $d_{P(G)}(u) = d_G(u) + d_G(v)$.
6. The ***quasivertex-total graph*** $Q(G)$ as the graph [7] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if (i) they are adjacent vertices of G or (ii) they are nonadjacent vertices of G (iii) they are adjacent edges of G or (iv) one is a vertex of G and other is an edge of G incident with it. If u is a vertex of G, then $d_{Q(G)}(u) = n - 1 + d_G(u)$. If $e = uv$ is an edge of G, then $d_{Q(G)}(e) = d_G(u) + d_G(v)$.

In Figure 1 self-explanatory examples of these derived graphs are depicted.

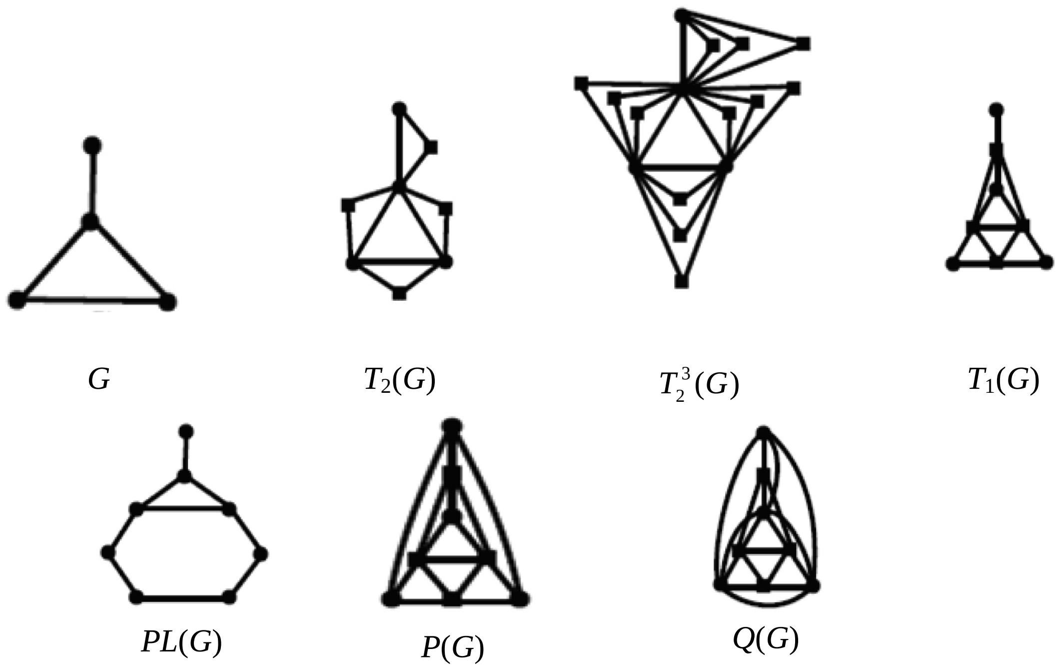

Figure 1. Various graphs derived from the graph *G* and $T_2^3(G)$ is *k-th* semitotal-point graph of *G* for *k* = 3.

The vertices of derived graphs depicted in Figure 1 except from the paraline graph *PL*, corresponding to the vertices of the parent graph *G*, are indicated by circles. The vertices of these graphs corresponding to the edges of the parent graph *G* are indicated by squares. In this paper we obtain neighbourly irregular derived graphs.

Theorem 2.1 [12] Let G be a graph. The subdivision graph S(G) is NI if and only if G does not have any vertex of degree two.

Theorem 2.2 [12] For any graph G, its line graph L(G) is NI graph if and only if N(u) contains all vertices of different degree for all $u \in V(G)$.

Theorem 2.3 [2] If G is NI graph, then $\overline{G}$ is not NI graph.

Theorem 2.4 [12] If G is NI graph, then L(G) is not NI graph.

Theorem 2.5 [12] For each integer $k \geq 1$, there exist a graph G with maximum degree $\Delta(G) = k$ such that L(G) is NI graph.

3. RESULTS

Theorem 3.1 For any graph G, the semitotal-point graph $T_2(G)$ is NI if and only if G is NI graph and no vertex of degree one is in G.

Proof. Suppose G is NI graph and no vertex of degree one is in G. In $T_2(G)$, let e = xy be an edge. Then x, y ∈ V(G) or x ∈ V(G) and y ∈ E(G).

(a) x, y ∈ V(G). Since $d_G(x) \neq d_G(y)$, $d_{T_2(G)}(x) = 2d_G(x) \neq 2d_G(y) = d_{T_2(G)}(y)$.

(b) x ∈ V(G) and y ∈ E(G). Since no vertex of degree is one in G and $d_{T_2(G)}(y) = 2$, $d_{T_2(G)}(x) = 2d_G(x) \neq 2 = d_{T_2(G)}(y)$. Thus from all the cases $T_2(G)$ is NI graph.

Conversely, suppose G is not NI graph. Then $d_G(x) = d_G(y)$ for some vertices x and y are adjacent in G. So, $d_{T_2(G)}(x) = d_{T_2(G)}(y)$. A contradiction to $T_2(G)$ is NI graph. Suppose $d_G(v) = 1$ for some v ∈ V(G). Let e = vy be an edge in $T_2(G)$. Then $d_{T_2(G)}(v) = 2d_G(v) = 2 = d_{T_2(G)}(y)$. Again a contradiction to $T_2(G)$ is NI graph. □

Theorem 3.2 For any graph G, the k^{th} semitotal-point graph is NI if and only if G is NI graph and $k \geq 2$.

Proof. The proof of this theorem is similar to the proof of the Theorem 3.1, so is omitted. □

Theorem 3.3 For any graph G, its $T_1(G)$ is NI if and only if L(G) is NI graph.

Proof. Suppose L(G) is NI graph. In $T_1(G)$, let e = xy be an edge. Then x, y ∈ E(G) or x ∈ V(G) and y ∈ E(G).

(a) x, y ∈ E(G). Let $x = v_iv_j$ and $y = v_iv_k$, so that x and y are adjacent in $T_1(G)$. Since L(G) is NI graph, we have $d_{L(G)}(x) \neq d_{L(G)}(y)$, $d_G(v_i) + d_G(v_j) - 2 \neq d_G(v_i) + d_G(v_k) - 2$ or $d_G(v_i) + d_G(v_j) \neq d_G(v_i) + d_G(v_k)$. Therefore $d_{T_1(G)}(x) \neq 2d_{T_1(G)}(y)$.

(b) x ∈ V(G) and y ∈ E(G). Let $e = xy = v_ie_j$ for some $v_i \in V(G)$ and $e_j \in E(G)$. Therefore $d_{T_1(G)}(x) = d_{T_1(G)}(v_i) = d_G(v_i)$ and $d_{T_1(G)}(y) = d_{T_1(G)}(e_j) = d_G(v_i) + d_G(v_k)$ where $e_j = v_iv_k \neq d_G(v_i)$ as $d_G(v_k) \neq 0 = d_G(x) = d_{T_1(G)}(x)$. Therefore for every pair of adjacent vertices in $T_1(G)$ have different degree. Thus $T_1(G)$ is NI graph.

Conversely, suppose L(G) is not NI graph. Then $d_{L(G)}(e_i) = d_{L(G)}(e_j)$ for some $e_i = v_rv_s$ and $e_j = v_rv_k$ are adjacent vertices in L(G). Hence, $d_G(v_r) + d_G(v_s) - 2 = d_G(v_r) + d_G(v_k) - 2$, $d_G(v_r) + d_G(v_s) = d_G(v_r) + d_G(v_k)$. Therefore $d_{T_1(G)}(e_i) = d_{T_1(G)}(e_j)$. A contradiction to $T_1(G)$ is NI graph. □

From Theorems 2.4, 2.5 and 3.3, we have the following corollaries.

Corollary 3.4 If G is NI graph, then $T_1(G)$ is not NI graph.

Corollary 3.5 For each integer $k \geq 1$, there exists a graph G with maximum degree $\Delta(G) = k$ such that $T_1(G)$ is NI graph.

Theorem 3.6 For any graph $G \neq K_2$, the paraline graph PL(G) is not NI graph.

Proof. Let v be a vertex of degree at least two in G. Then neighbourhood of v in S(G) has at least two vertices of degree two. By Theorem 2.2, L(S(G))=PL(G) is not NI graph. □

Theorem 3.7. For any graph $G \neq K_2$, the quasi-total graph P(G) is not NI graph.

Proof. Let $G \neq K_2$ be a graph. We have the following cases:

Case 1. If G is not a complete graph, then there exist at least two vertices $u, v \in V(G)$ such that $d_{P(G)}(u) = d_{P(G)}(v) = n - 1$. Therefore P(G) is not NI graph.

Case 2. If G is a complete graph, then there exist at least two edges $e_i, e_j \in E(G)$ such that $d_{P(G)}(e_i) = d_{P(G)}(e_j)$. Therefore P(G) is not NI graph. □

Theorem 3.8 For any graph G with n vertices, the quasivertex-total graph Q(G) is NI if and only if G, $\overline{G}$ and L(G) all are NI graphs and $\Delta(G) \neq n - 1$.

Proof. Suppose G, $\overline{G}$ and L(G) all are NI graphs. In Q(G), let e = xy be an edge, then $x, y \in V(G)$ or $x, y \in V(\overline{G})$ or $x, y \in E(G)$ or $x \in V(G)$ and $y \in E(G)$.

(a) $x, y \in V(G)$. Since $d_G(x) \neq d_G(y)$, $d_{Q(G)}(x) = n - 1 + d_G(x) \neq n - 1 + d_G(y) = d_{Q(G)}(y)$.

(b) $x, y \in V(\overline{G})$. Since $d_{\overline{G}}(x) \neq d_{\overline{G}}(y)$, $d_{Q(G)}(x) = n - 1 + d_G(x) \neq n - 1 + d_G(y) = d_{Q(G)}(y)$.

(c) $x, y \in E(G)$. Let $x = v_iv_j$ and $y = v_iv_k$. So that x and y are adjacent in Q(G). Therefore $d_{Q(G)}(x) = d_G(v_i) + d_G(v_j)$ and $d_{Q(G)}(x) = d_G(v_i) + d_G(v_k)$. But $d_{L(G)}(x) \neq d_{L(G)}(y)$ as L(G) is NI graph, $d_{L(G)}(x) = d_G(v_i) + d_G(v_j) - 2$ and $d_{L(G)}(y) = d_G(v_i) + d_G(v_k) - 2$. Therefore $d_{Q(G)}(x) \neq d_{Q(G)}(y)$.

(d) $x \in V(G)$ and $y \in E(G)$. Let $e = xy = v_ie_j$ for some $v_i \in V(G)$ and $e_j \in E(G)$. Then $d_{Q(G)}(y) = d_{Q(G)}(e_j) = d_{L(G)}(e_j) + 2$ where $e_j = v_iv_j = d_G(v_i) + d_G(v_j) \neq n - 1 + d_G(v_i)$ as $\Delta(G) \neq n - 1 \neq d_{Q(G)}(x)$. Thus in all the cases Q(G) is NI graph.

Conversely, suppose Q(G) is NI graph. We have to prove that G, $\overline{G}$ and L(G) are all NI graphs. If G is not NI graph, then there exists an edge $e_k = v_iv_j$ in G such that $d_G(v_i) = d_G(v_j)$. Therefore $n - 1 + d_G(v_i) = n - 1 + d_G(v_j)$. So, $d_{Q(G)}(v_i) = d_{Q(G)}(v_j)$. A contradiction to Q(G) is NI graph. Suppose $\overline{G}$ is not NI graph, then there exists an edge $e_k = v_iv_j$ in $\overline{G}$ such that $d_{\overline{G}}(v_i) = d_{\overline{G}}(v_j)$. Therefore $n - 1 + d_G(v_i) = n - 1 + d_G(v_j)$ and so $d_{Q(G)}(v_i) = d_{Q(G)}(v_j)$. A contradiction to Q(G) is NI graph.

Suppose L(G) is not NI graph, then there exists two adjacent vertices $e_i = v_rv_s$ and $e_j = v_rv_k$ in L(G) with $d_{L(G)}(e_i) = d_{L(G)}(e_j)$. Thus $d_G(v_r) + d_G(v_s) - 2 = d_G(v_r) + d_G(v_k) - 2$. Hence $d_G(v_r) + d_G(v_s) = d_G(v_r) + d_G(v_k)$ and so $d_{Q(G)}(e_i) = d_{Q(G)}(e_j)$. Again a contradiction to Q(G) is NI graph. Suppose $\Delta(G) = n - 1 = d_G(v)$ and let $e = uv$ be an edge. Then $d_{Q(G)}(e) = d_{Q(G)}(u)$. Again a contradiction to Q(G) is NI graph. □

From Theorems 2.3, 2.4 and 3.8 we have following result.

Theorem 3.9 There is no nontrivial graph G whose quasivertex-total graph Q(G) is NI graph.

4. NEIGHBOURLY IRREGULAR GRAPH PRODUCTS

The corona [10] of two graphs G and H is the graph obtained by taking one copy of G, |V(G)| copies of H and joining each i-th vertex of G to every vertex in the i-th copy of H. The edge corona [5] of two graphs G and H denoted by $G \lozenge H$ is obtained by taking one copy of G and |E(G)| copies of H and joining each end vertices of i-th edge of G to every vertex in the i-th copy of H.

Theorem 4.1 Let G and H be nontrivial graphs. Then $G \lozenge H$ is NI graph if and only if both G and H are NI graphs and, G does not have pendent vertex or $\Delta(H) < |V(H)| - 1$, where $\Delta(H)$ is the maximum degree of the vertices of H.

Proof. To prove the result, we have to present some notations. Let G′ be the copy of G and H_i be the i-th copy of H in $G \lozenge H$, $1 \leq i \leq |E(G)|$. A vertex of $G \lozenge H$ corresponding to the vertex u in H is denoted by u′. Also, we denote a vertex of $G \lozenge H$ corresponding to the vertex v in G by v′.

Let G and H be NI graphs and, G does not have pendent vertex or $\Delta(H) < |V(H)| - 1$. Then it is clear that $G \lozenge H$ is NI graph.

Conversely, let G and H be two nontrivial graphs and $G \lozenge H$ is NI graph. Suppose $u'v' \in E(G \lozenge H)$ such that $u', v' \in V(H_i)$, then $d_{G\lozenge H}(u') - d_{G\lozenge H}(v') = d_H(u) - d_H(v) \neq 0$ and so H is NI graph. Also, if $u'v' \in E(G \lozenge H)$ such that $u', v' \in V(G')$, then $d_{G\lozenge H}(u') - d_{G\lozenge H}(v') = (|V(H)| + 1)(d_G(u) - d_G(v)) \neq 0$. Thus, G is NI graph. On the other hand, if $u'v' \in E(G \lozenge H)$ such that $u' \in V(G')$, and $v' \in V(H_i)$, then $d_{G\lozenge H}(u') - d_{G\lozenge H}(v') = (|V(H)| + 1)$

$d_G(u) - (d_H(v) + 2) \neq 0$ and it shows that, G does not have pendent vertex or $\Delta(H) < |V(H)| - 1$. □

To present the next results, we need two definitions as follows: The cluster G{H} is obtained by taking one copy of G and |V(G)| copies of a rooted graph H, and by identifying the root of the i-th copy of H with the i-th vertex of G, i = 1, 2, …, |V(G)| [11].

Suppose G and H are graphs with disjoint vertex sets. Following Došlić [3], for given vertices $y \in V(G)$ and $z \in V(H)$ a splice of G and H by vertices y and z, $(G \cdot H)(y, z)$, is defined by identifying the vertices y and z in the union of G and H.

Theorem 4.2 Let G and H be graphs. Then G{H} is NI graph if and only if both G and $(H \cdot S_{d_G(u_i)})(r, x)$ are NI graphs, for each i = 1, 2, …, |V(G)|, where x is the vertex with maximum degree of the star $S_{d_G(u_i)}$ and r the root vertex of H.

Proof. Let G and $(H \cdot S_{d_G(u_i)})(r, x)$ be NI graphs, for each i = 1, 2, …, |V(G)|, where x is the vertex with maximum degree of the star $S_{d_G(u_i)}$ and r the root vertex of H. Then, it is clear that G{H} is NI graph.

Conversely, let G{H} be NI graph. Also, suppose $u'v' \in E(G\{H\})$ and u′, v′ are the vertices of G{H} corresponding to the vertices u, v in G, respectively. If u′ and v′ are vertices of a copy of G, then $d_{G\{H\}}(u') - d_{G\{H\}}(v') = d_G(u) - d_G(v) \neq 0$. So G is NI graph. On the other hand, suppose $u'v' \in E(G\{H\})$ and u′, v′ are the vertices of $G\{H\} \cap H_i$ corresponding to the vertices u, v in H, respectively. Then, it is not difficult to see that $dG\{H\}(u') - dG\{H\}(v') \neq 0$ if and only if

$$d\left(H \cdot S_{d_G(u_i)}\right)_{(r,x)}(u) - d\left(H \cdot S_{d_G(u_i)}\right)_{(r,x)}(v) \neq 0 .$$

So, $(H \cdot S_{d_G(u_i)})(r, x)$ is NI graph. □

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