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CHARACTERIZATION OF TOTAL LINE-CUT GRAPHS

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Abstract. The total line-cut graph of a graph $G = (V, E)$, denoted by $TL_c(G)$, is the graph with point set $E(G) \cup W(G)$, where $W(G)$ is the set of cutpoints of G, in which two points are adjacent if and only if they correspond to adjacent lines of G or correspond to adjacent or coadjacent cutpoints of G or one point corresponds to a line e of G and the other corresponds to a cutpoint c of G such that e is incident with c. In this paper, we offer a structural characterization of total line-cut graphs.

1. Introduction

By a graph G=(V, E), we mean a finite, undirected graphs without loops or multiple lines. For any graph G, let $V(G)$, $E(G)$, $W(G)$ and $U(G)$ denote the point set, line set, cutpoint set and block set of G, respectively. The lines and cutpoints of a graph are called its members. A *pendant point* is a point of degree one and a line incident (nonincident) with a pendant point is called *pendant (nonpendant) line*. The *neighborhood* of a point u in V is the set $N(u)$ consisting of all points v which are adjacent with u. A *cutpoint* of a connected graph G is the one whose removal increases the number of components. A *nonseparable graph* is connected, nontrivial and has no cutpoints. A *block* of a graph G is a maximal nonseparable subgraph. A block is called *pendantblock* of a graph if it contains exactly one cutpoint of G. The *line graph* $L(G)$ of G is the graph whose point set is $E(G)$ in which two points are adjacent if and only if they are adjacent in G. If $B = \{u_1, u_2, ..., u_n; n \geq 2\}$ is a block of G, then we say that point u_1 and block B are incident with each other, as are u_2 and B and so on. If a block is incident with cutpoints $c_1, c_2, ..., c_r$, $r \geq 2$, we say that c_i and c_j are *coadjacent* where $i \neq j$ and $1 \leq i, j \leq r$. The *cutpoint graph* $C(G)$ of a graph G is the graph whose point set corresponds to the cutpoints of G and in which two points of $C(G)$ are adjacent if the cutpoints of G to which they correspond lie on a common block [3]. For graph theoretic terminology, we refer to [3, 5].

Kulli and Muddebihal [4] introduced the idea of a lict graph and litact graph. In [1], M. Acharya et al. called lict graph as a line-cut graph and gave the characterization of line-cut graph. In [2], we called litact graph as a total line-cut

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graph, now we give the characterization of total line-cut graph.

Definition 1.1. The *total line-cut graph* (also known as litact graph) of a graph $G = (V, E)$, denoted by $TL_c(G)$, is the graph with point set $E(G) \cup W(G)$, where $W(G)$ is the set of cutpoints of G, in which two points are adjacent if and only if they correspond to adjacent lines of G or correspond to adjacent or coadjacent cutpoints of G or one point corresponds to a line e of G and the other corresponds to a cutpoint c of G such that e is incident with c.

Figure 1. illustrates a graph and its total line-cut graph.

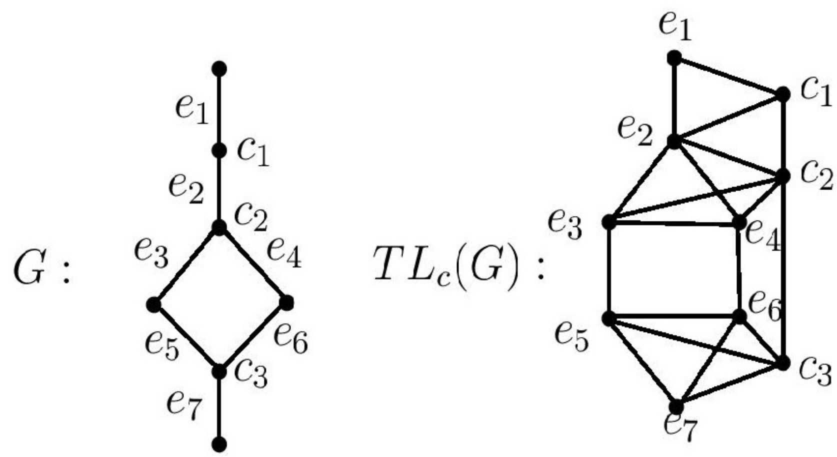

Figure 1. A graph G and its total line-cut graph $TL_c(G)$.

The point c_i (e_i) of total line-cut graph $TL_c(G)$ corresponding to a cutpoint c_i (line e_i) of G and is referred to as cutpoint (line) vertex.

2. Main result

A graph G is a total line-cut graph if and only if it is isomorphic to the total line-cut graph $TL_c(H)$ of some graph H. Let G=(V, E) be a graph and let $V' \subseteq V(G)$. The induced subgraph $< V' >$ of G is called a *clique* of G if $< V' >$ is isomorphic with a complete graph of order $|V'|$. A *clique is maximal* if it is not a subgraph of a clique of larger order.
The following theorem gives the characterization of total line-cut graph.

Theorem 2.1. *The following statements are equivalent:*

(I) *$G = (V, E)$ is a total line-cut graph, i.e $G \cong TL_c(H)$ of some graph H.*

(II) *The lines of G can be partitioned among the three types of maximal cliques, namely; maximal cliques induced by the line vertices of G, maximal cliques induced by the cutpoint vertices of G and maximal cliques induced by the line vertices and cutpoint vertices of G, satisfying following conditions;*

(1) *In the maximal cliques G_i which are induced by the line vertices of G, no point lies in more than two maximal cliques and for each clique G_i in the partition*

  (a) *if each point of G_i lies in two cliques of the partition, then $G - E(G_i)$ is connected and*

(b) *if all but one point, v, of G_i lies in two cliques of the partition, then $G - E(G_i) - v$ disconnected. (that is , G does not contain a pendant point.)*

(2) *In the maximal cliques G_i which are induced by the cutpoint vertices or line vertices and cutpoint vertices of G*

(a) *no line vertex of G lies in more than two maximal cliques*

(b) *if cutpoint vertex c_i lies in one or two (more than two)cliques, then corresponding cutpoint c_i lies on pendantblock (nonpendantblock)of H.*

Proof. $(I) \Longrightarrow (II)$
Let G ba a total line-cut graph. Therefore $G \cong TL_c(H)$ for some graph H. We assume that G has no isolated points. By definition of total line-cut graph, the lines incident on a point v of H with degree $deg(v) = p$, that is not a cutpoint, induces a maximal clique of G with order p. The lines incident on a cutpoint c of H with $deg(v) = p$ induces a maximal clique of G with order $p + 1$ having c as one of its points. Let $U(G)$=$\{B_1, B_2, ..., B_n\}$, $n \geq 2$ be the block set of G and $C(B_i)$ be the number of cutpoints of a connected graph G which are the points of the block B_i. Then cutpoint vertices induces a maximal clique of order $1 + \sum_{i=1}^{n}(C(B_i) - 1)$. Because every line of G either results from two adjacent lines of H or from a cutpoint of H and a line of H that is incident with that cutpoint or adjacent or coadjacent cutpoints of H, then every line of G is contained in precisely one such clique. This is illustrated in Figure 2.

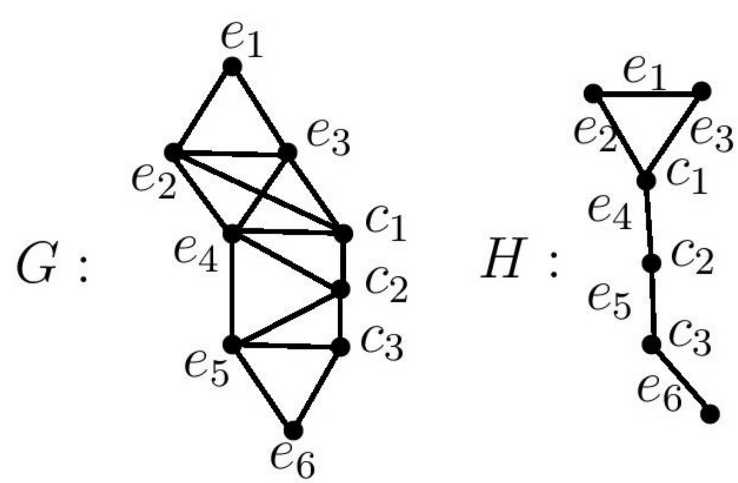

Figure 2. A graph G and a graph H such that $G \cong TL_c(H)$.

Note that $V(G) = E(H) \cup W(H)$, where $W(H)$ is the set of cutpoints of H. Clearly e_i is a pendant line of H then the corresponding line vertex in G is contained in only one maximal clique. If e_i is a nonpendant line of H, then the corresponding line vertex in G is contained in precisely two maximal cliques. Therefore no line vertex of G lies in more than two maximal cliques. Also if cutpoint of H lies on pendantblock, then the corresponding cutpoint vertex in

G is contained in one or two maximal cliques. If cutpoint of H lies on nonpendantblock, then the corresponding cutpoint vertex in G is contained in more than two maximal cliques. Thus the lines of G can be partitioned among the maximal cliques of G in such a way that no line vertex of G lies in more than two maximal cliques and each cutpoint vertex lies in more than one maximal cliques.

In the maximal clique G_i induced by the line vertices of G, if each point of G_i is contained in two maximal cliques, then G_i is induced by the lines incident with a noncutpoint, v, of H. Suppose a point v_j of G_j is also contained in maximal clique G_j. Then the maximal clique G_j must result from points of H belongs to $N(v)$, the neighborhood of v in H. Because $H - v$ is connected, then $G - E(G_i)$ is connected. Therefore Condition $1(a)$ is satisfied.

Figure 3. illustrates a graph G that does not satisfy Condition 1(a) and is not a total line-cut graph.

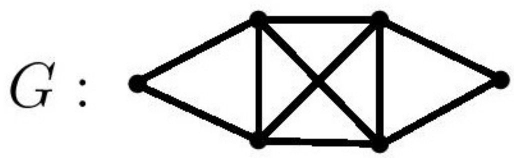

Figure 3. A graph G that is not a total line-cut graph.

In the maximal clique G_i induced by the line vertices of G, if all but one point, v, of G_i is not lies in two cliques of the partition, then all points of G_i lies in only one maximal cliques. Because $H - v$ is connected, then $G - E(G_i) - v$ is connected, a contradiction. Therefore Condition $1(b)$ is satisfied.

Figure 4. illustrates graphs that do not satisfy Condition 1(b) and are not total line-cut graphs.

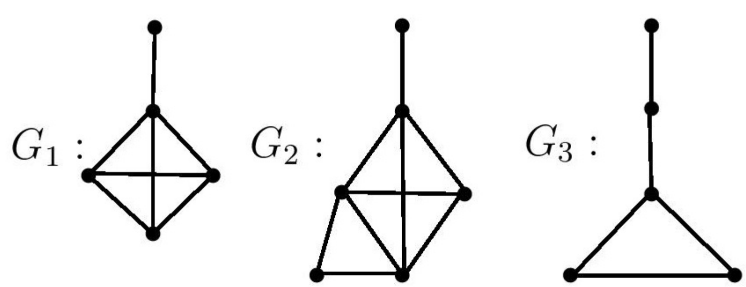

Figure 4. Graphs that are not total line-cut graphs.

Since if e_i is a pendant line of H, then the corresponding line vertex in G is contained in only one maximal clique. If e_i is a nonpendant line of H, then the corresponding line vertex in G is contained in precisely two maximal cliques. Therefore Condition 2(a) is satisfied.

Figure 5. illustrates a graph G that does not satisfy Condition 2(a) and is not a total line-cut graph.

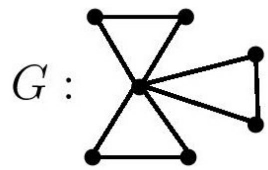

Figure 5. A graph G that is not a total line-cut graph.

If H has only one cutpoint, then corresponding cutpoint vertex lies on only one maximal clique. Suppose H has more than one cutpoint. If cutpoint lies on pendantblock of H, then corresponding cutpoint vertex lies in maximal clique induced by the line vertices and cutpoint vertices and the maximal clique induced by the cutpoint vertices of G. Therefore cutpoint vertices lies in one or two maximal cliques. And if cutpoint lies on nonpendantblock of H, then the corresponding cutpoint vertex contained in at least three maximal cliques of G. Therefore Condition 2(b) is satisfied.

Figure 6. illustrates a graph G that does not satisfy Condition 2(b) and is not a total line-cut graph.

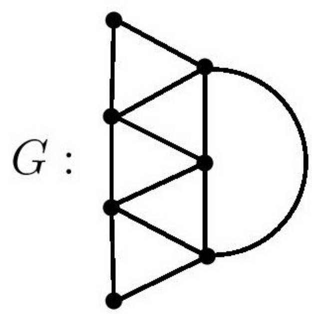

Figure 6. A graph G that is not a total line-cut graph.

$(II) \Longrightarrow (I)$

Let $P(G)$=$\{G_1, G_2, ..., G_n\}$ be set of cliques of G induced by the line vertices of G, $C(G)$=$\{c_1, c_2, ..., c_n\}$ be the set of cutpoint vertices of G. We provide the construction of a graph H whose total line-cut graph is G. The point set of H is the union of sets $P(G)$, $C(G)$ and the set U of line vertices of G that belongs to only one of the cliques G_i. Thus, $V(H) = P(G) \cup C(G) \cup U$ and two of these points are adjacent whenever the point belongs to set U and the point belongs to set $C(G)$ are adjacent or the points of the cliques G_i belongs to set $P(G)$ and the point belongs to set $C(G)$ are adjacent or two cliques of the set $P(G)$ are incident with a common point or two cutpoints are adjacent if the cliques induced by the corresponding cutpoint vertices and line vertices have a point in common. For this graph H, $G \cong TL_c(H)$. Hence , G is a total line-cut graph.

Figures 7 and 8. illustrates construction of a graph H such that $G \cong TL_c(H)$ for a graph G that satisfies Condition (II).

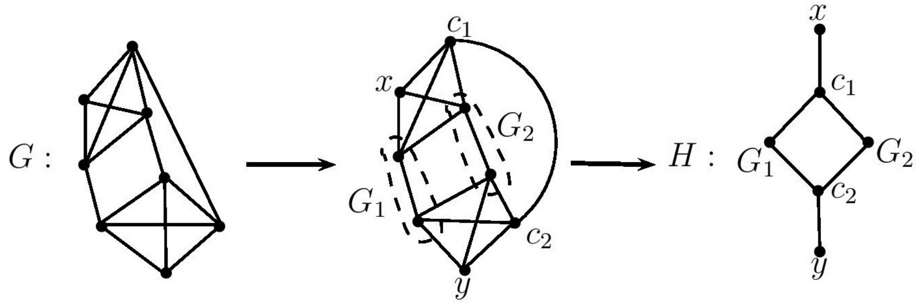

Figure 7. A graph G and a graph H such that $G \cong TL_c(H)$.

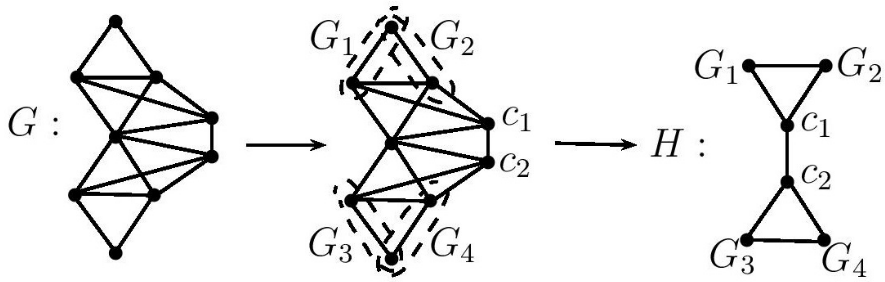

Figure 8. A graph G and a graph H such that $G \cong TL_c(H)$.

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