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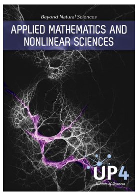

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Computing First Zagreb index and F-index of New C-products of Graphs

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Abstract

For a (molecular) graph, the first Zagreb index is equal to the sum of squares of the degrees of vertices, and the F-index is equal to the sum of cubes of the degrees of vertices. In this paper, we introduce sixty four new operations on graphs and study the first Zagreb index and F-index of the resulting graphs.

Keywords: Zagreb indices, forgotten topological index, hyper-Zagreb index, C-products of graphs.
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1 Introduction

Throughout this paper, we consider only simple graphs. Let G be such a graph on n vertices and m edges. We denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. Thus, $|V(G)| = n$ and $|E(G)| = m$. As usual, n is said to be the order and m the size of G. If u and v are two adjacent vertices of G, then the edge connecting them will be denoted by uv. The degree of a vertex $w \in V(G)$ is the number of vertices adjacent to w and is denoted by $d_G(w)$. The complement of G, denoted by $\overline{G}$, is a graph which has the same vertex set as G, in which two vertices are adjacent if and only if they are not adjacent in G. The line graph $L(G)$ of a graph G is the graph with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G have a vertex in common. The subdivision graph $S(G)$ of a graph G whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and other is an edge of G incident with it. The

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partial complement of subdivision graph $\overline{S}(G)$ of a graph G whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and the other is an edge of G non incident with it. Please refer to [17, 25] for unexplained graph theoretic terminology and notation.

In theoretical chemistry, the physico-chemical properties of chemical compounds are often modeled by means of molecular-graph-based structure-descriptors which are also referred to topological indices [16, 30]. Topological indices are found to be very useful in chemistry, biochemistry and nanotechnology in isomer discrimination, structure-property relationship, structure-activity relationship and pharmaceutical drug design. The first and second Zagreb indices of a graph are among the most studied vertex degree based topological indices. The first and second Zagreb indices, respectively defined by

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 = \sum_{uv \in E(G)} [d_G(u) + d_G(v)] \text{ and } M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v)$$

are widely studied degree-based topological indices, that were introduced by Gutman and Trinajstić [15] in 1972.

The vertex-degree-based graph invariant

$$F(G) = \sum_{v \in V(G)} d_G(v)^3 = \sum_{uv \in E(G)} [d_G(u)^2 + d_G(v)^2]$$

was encountered in [15]. Recently there has been some interest to F, called forgotten topological index or F-index [10].

Shirdel et al. [29] introduced a new Zagreb index of a graph G named hyper-Zagreb index and is defined as:

$$HM(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))^2.$$

Computation of these topological indices of graphs are reported in [2–4, 11–13].

Li and Zhao [27] introduced the first general Zagreb index as follows

$$\alpha_\lambda(G) = \sum_{u \in V(G)} [d_G(u)]^\lambda.$$

It is easy to write that

$$\alpha_\lambda(G) = \sum_{uv \in E(G)} [(d_G(u))^{\lambda-1} + (d_G(v))^{\lambda-1}].$$

The general sum connectivity index [31] was introduced by Zhou et al. and is defined as

$$M_1^\alpha(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^\alpha. \tag{1}$$

By Eq. (1), it is consistent to define $M_1^3(G)$ as

$$M_1^3(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^3.$$

Here we note that, $\alpha_2(G) = M_1^1(G) = M_1(G)$, $\alpha_3(G) = F(G)$ and $M_1^2(G) = HM(G)$.

Graph operations play a vital role in chemical graph theory. Different chemically important graphs can be obtained by applying graph operations on some general or particular graphs. One of the chemically interseting graph operation is Cartesian product of graphs. The Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(u_1, v_1)(u_2, v_2)$ is an edge of $G_1 \times G_2$ if and only if $[u_1 = u_2$ and $v_1 v_2 \in E(G_2)]$ or $[v_1 = v_2$ and $u_1 u_2 \in E(G_1)]$.

Many of the chemically interesting graphs can be obtained by applying the Cartesian product of graphs. For example, the ladder graph L_n is the molecular graph related to the polynomial structure obtained by the Cartesian product of P_2 and P_{n+1}. The C_4 nanotube $TUC_4(m,n)$ is the Cartesian product of P_n and P_m and the C_4 nanotorus $TC_4(m,n)$ is the Cartesian product of C_n and C_m.

Graovac and Pisanski [14] were the first to consider the problem of computing topological indices of graph operations. In their paper, they computed an exact formula for the Wiener index of the Cartesian product of graphs. In [24], Klavzar, Rajapakse and Gutman computed the Szeged index of the Cartesian product graphs. In a series of recent papers [18–23], M. H. Khalifeh and his coworkers extended this program to other topological indices, such as the vertex and edge PI index, the first and second Zagreb index, the vertex and edge versions of Szeged index, and the hyper-Wiener and the edge-Wiener indices of several operations. The present work is the continuation of research along the same lines, and is concerned with additional types of graph operations.

2 New Cartesian products of graphs

Eliasi et al. in [9] generalized the concept of Cartesian products of graphs, and introduced four new sums of graphs called F-sums of graphs and studied the Wiener index of resulting graphs. Recently there has been some interest on computing topological indices of F-sums of graphs [1, 5, 8, 26, 28].

Motivated by applications of Cartesian product of graphs, here we are more generalize the concept of Cartesian products of graphs and introduce the new C-products of graphs. For this purpose we proceed to introduce some notions and definition of [7].

For a graph $G = (V,E)$, let G^0 be the graph with $V(G^0) = V(G)$ and with no edges, G^1 the complete graph with $V(G^1) = V(G)$, $G^+ = G$, and $G^- = \overline{G}$.

Definition 1. [7] Given a graph G with vertex set $V(G)$ and edge set $E(G)$ and three variables $x,y,z \in \{0,1,+,-\}$, the xyz-transformation graph $T^{xyz}(G)$ of G is the graph with vertex set $V(T^{xyz}(G)) = V(G) \cup E(G)$ and the edge set $E(T^{xyz}(G)) = E((G)^x) \cup E((L(G))^y) \cup E(W)$ where $W = S(G)$ if $z = +$, $W = \overline{S}(G)$ if $z = -$, W is the graph with $V(W) = V(G) \cup E(G)$ and with no edges if $z = 0$ and W is the complete bipartite graph with parts $V(G)$ and $E(G)$ if $z = 1$.

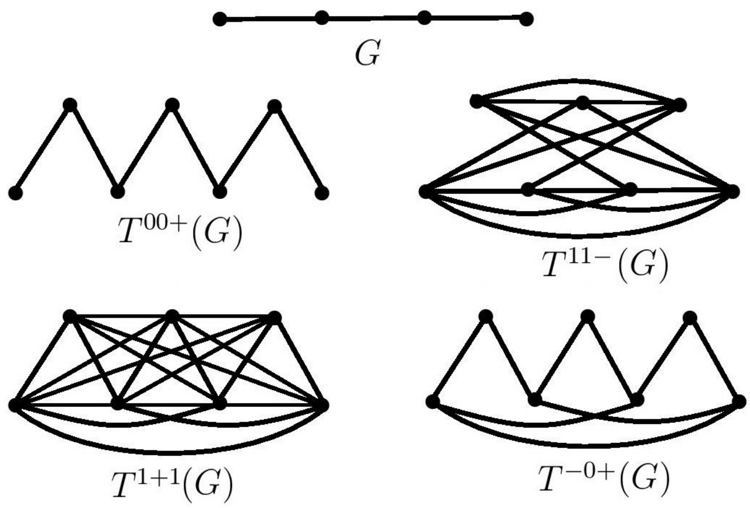

Fig. 1 Some $xyz-$transformations of a 4-vertex path.

Examples of $xyz-$transformations of a 4-vertex path are given in Figure 1. We call vertex in xyz-transformation graphs corresponding to vertex of parent graph as point vertex whereas vertex in xyz-transformation graphs corresponding to edge of parent graph as line vertex.

Now we give the definition of the C-product of graphs in the following.

Definition 2. Let $C \in \{T^{xyz} | x,y,z \in \{0,1,+,-\}\}$. The C-product of G_1 and G_2, denoted by $G_1 \times_C G_2$, is a graph with the set of vertices $V(G_1 \times_C G_2) = (V(G_1) \cup E(G_1)) \times V(G_2)$ and two vertices (u_1,u_2) and (v_1,v_2) of $G_1 \times_C G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(C(G_1))]$.

Thus we obtain 64 new $C-$ products of graphs in which $G_1 \times_{T^{00+}} G_2$, $G_1 \times_{T^{+0+}} G_2$, $G_1 \times_{T^{0++}} G_2$ and $G_1 \times_{T^{+++}} G_2$ are $F-$ sums of graphs introduced by Eliasi and Taeri [9]. Examples of C-products of P_4 and P_2 are given in Figure 2. In this paper, we compute the expressions for first Zagreb index and F-index of the $C-$ products of graphs.

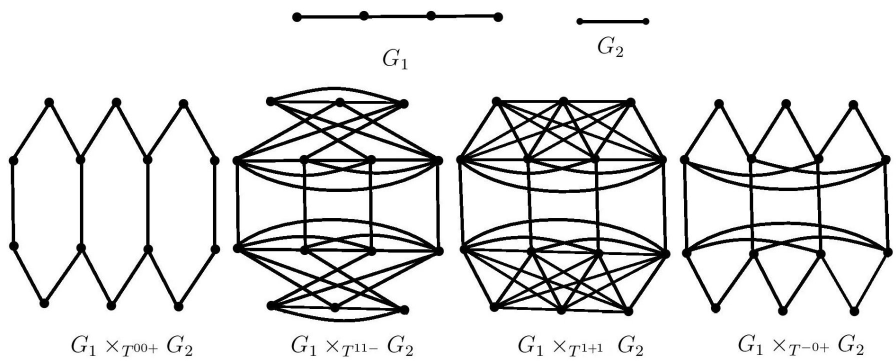

Fig. 2 Some C-product of P_4 and P_2.

3 Main Results

We start by stating the following propositions, which are immediately from definitions and needed for the proving our main results.

Proposition 1. *Let G be a (n,m)-graph. Then the degree of point vertex u and line vertex $e(= ab$ in $G)$ in $T^{xyz}(G)$ when $z = 0$ are*

(i) $$d_{T^{xy0}(G)}(u) = \begin{cases} 0 & if\ x = 0\ \&\ y \in \{0,1,+,-\}. \\ n-1 & if\ x = 1\ \&\ y \in \{0,1,+,-\}. \\ d_G(u) & if\ x = +\ \&\ y \in \{0,1,+,-\}. \\ n-1-d_G(u) & if\ x = -\ \&\ y \in \{0,1,+,-\}. \end{cases}$$

(ii) $$d_{T^{xy0}(G)}(e) = \begin{cases} 0 & if\ y = 0\ \&\ x \in \{0,1,+,-\}. \\ m-1 & if\ y = 1\ \&\ x \in \{0,1,+,-\}. \\ d_G(a)+d_G(b)-2 & if\ y = +\ \&\ x \in \{0,1,+,-\}. \\ m+1-d_G(a)-d_G(b) & if\ y = -\ \&\ x \in \{0,1,+,-\}. \end{cases}$$

Proposition 2. *Let G be a (n,m)-graph. Then the degree of point vertex u and line vertex $e(= ab$ in $G)$ in $T^{xyz}(G)$ when $z = 1$ are*

(i) $$d_{T^{xy1}(G)}(u) = \begin{cases} m & if\ x = 0\ \&\ y \in \{0,1,+,-\}. \\ m+n-1 & if\ x = 1\ \&\ y \in \{0,1,+,-\}. \\ d_G(u)+m & if\ x = +\ \&\ y \in \{0,1,+,-\}. \\ n-1-d_G(u)+m & if\ x = -\ \&\ y \in \{0,1,+,-\}. \end{cases}$$

(ii) $d_{T^{xy1}(G)}(e) = \begin{cases} n & if\ y = 0\ \&\ x \in \{0,1,+,-\}. \\ n+m-1 & if\ y = 1\ \&\ x \in \{0,1,+,-\}. \\ n-2+d_G(a)+d_G(b) & if\ y = +\ \&\ x \in \{0,1,+,-\}. \\ n+m+1-d_G(a)-d_G(b) & if\ y = -\ \&\ x \in \{0,1,+,-\}. \end{cases}$

Proposition 3. *Let G be a (n,m)-graph. Then the degree of point vertex u and line vertex $e(= ab$ in $G)$ in $T^{xyz}(G)$ when $z = +$ are*

(i) $d_{T^{xy+}(G)}(u) = \begin{cases} d_G(u) & if\ x = 0\ \&\ y \in \{0,1,+,-\}. \\ d_G(u)+n-1 & if\ x = 1\ \&\ y \in \{0,1,+,-\}. \\ 2d_G(u) & if\ x = +\ \&\ y \in \{0,1,+,-\}. \\ n-1 & if\ x = -\ \&\ y \in \{0,1,+,-\}. \end{cases}$

(ii) $d_{T^{xy+}(G)}(e) = \begin{cases} 2 & if\ y = 0\ \&\ x \in \{0,1,+,-\}. \\ m+1 & if\ y = 1\ \&\ x \in \{0,1,+,-\}. \\ d_G(a)+d_G(b) & if\ y = +\ \&\ x \in \{0,1,+,-\}. \\ m+3-d_G(a)-d_G(b) & if\ y = -\ \&\ x \in \{0,1,+,-\}. \end{cases}$

Proposition 4. *Let G be a (n,m)-graph. Then the degree of point vertex u and line vertex $e(= ab$ in $G)$ in $T^{xyz}(G)$ when $z = -$ are*

(i) $d_{T^{xy-}(G)}(u) = \begin{cases} m-d_G(u) & if\ x = 0\ \&\ y \in \{0,1,+,-\}. \\ n+m-1-d_G(u) & if\ x = 1\ \&\ y \in \{0,1,+,-\}. \\ m & if\ x = +\ \&\ y \in \{0,1,+,-\}. \\ n+m-1-2d_G(u) & if\ x = -\ \&\ y \in \{0,1,+,-\}. \end{cases}$

(ii) $d_{T^{xy-}(G)}(e) = \begin{cases} n-2 & if\ y = 0\ \&\ x \in \{0,1,+,-\}. \\ n+m-3 & if\ y = 1\ \&\ x \in \{0,1,+,-\}. \\ n+d_G(a)+d_G(b)-4 & if\ y = +\ \&\ x \in \{0,1,+,-\}. \\ n+m-1-d_G(a)-d_G(b) & if\ y = -\ \&\ x \in \{0,1,+,-\}. \end{cases}$

Proposition 5. *Let G_1 and G_2 be the graphs. If (u,v) is a vertex of $G_1 \times_C G_2$, then*

$$d_{G_1\times_C G_2}(u,v) = \begin{cases} d_{C(G_1)}(u)+d_{G_2}(v) & if\ u \in V(C(G_1)) \cap V(G_1),\ v \in V(G_2) \\ d_{C(G_1)}(u) & if\ u \in V(C(G_1)) \cap E(G_1),\ v \in V(G_2). \end{cases}$$

We are now prepared to state and prove our main results.

Theorem 6. *Let G_1 and G_2 be the graphs. Then*

$$\alpha_\lambda(G_1 \times_C G_2) = \sum_{u\in V(C(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} [d_{C(G_1)}(u)+d_{G_2}(v)]^\lambda + \sum_{v\in V(G_2)} \sum_{e\in V(C(G_1))\cap E(G_1)} d^\lambda_{C(G_1)}(e). \quad (2)$$

Proof. By the definition of first general Zagreb index , we have

$$\alpha_\lambda(G_1 \times_C G_2) = \sum_{(u,v)\in V(G_1\times_C G_2)} d^\lambda_{G_1\times_C G_2}(u,v)$$

We partition $V(G_1 \times_C G_2)$ into $V(C(G_1)) \cap V(G_1)$ and $V(C(G_1)) \cap E(G_1)$ and from Proposition 5, we have

$$\alpha_\lambda(G_1 \times_C G_2) = \sum_{u\in V(C(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} [d_{C(G_1)}(u)+d_{G_2}(v)]^\lambda + \sum_{v\in V(G_2)} \sum_{e\in V(C(G_1))\cap E(G_1)} d^\lambda_{C(G_1)}(e).$$

For $\lambda = 2,3$ in (2), we get the following equations.

$$M_1(G_1 \times_C G_2) = \sum_{u\in V(C(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} [d^2_{C(G_1)}(u) + d^2_{G_2}(v) + 2d_{C(G_1)}(u)d_{G_2}(v)] + \sum_{v\in V(G_2)} \sum_{e\in V(C(G_1))\cap E(G_1)} d^2_{C(G_1)}(e). \quad (3)$$

and

$$F(G_1 \times_C G_2) = \sum_{u\in V(C(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} [d^3_{C(G_1)}(u) + d^3_{G_2}(v) + 3d^2_{C(G_1)}(u)d_{G_2}(v) + 3d_{C(G_1)}(u)d^2_{G_2}(v)] + \sum_{v\in V(G_2)} \sum_{e\in V(C(G_1))\cap E(G_1)} d^3_{C(G_1)}(e). \quad (4)$$

For a given graph G_i, its vertex and edge sets will be denoted by $V(G_i)$ and $E(G_i)$, respectively, and their cardinalities by n_i and m_i, respectively, where $i = 1,2$.

By plugging the corresponding degrees of vertices of T^{xy0} from Proposition 1 in (3) and (4), bearing in mind that $\sum_{v\in V(G)} d_G(v) = 2m$, $\sum_{v\in V(G)} = n$ and $\sum_{e\in E(G)} = m$, we get the following two theorems.

Theorem 7. *Let G_1 and G_2 be the graphs. Then*

(1) $M_1(G_1 \times_{T^{000}} G_2) = n_1M_1(G_2)$

(2) $M_1(G_1 \times_{T^{100}} G_2) = n_1n_2(n_1-1)^2 + n_1M_1(G_2) + 4n_1m_2(n_1-1)$

(3) $M_1(G_1 \times_{T^{+00}} G_2) = n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2$

(4) $M_1(G_1 \times_{T^{-00}} G_2) = n_1n_2(n_1-1)^2 + n_2M_1(G_1) - 4n_2m_1(n_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1-1) - 8m_1m_2$

(5) $M_1(G_1 \times_{T^{010}} G_2) = n_1M_1(G_2) + n_2m_1(m_1-1)^2$

(6) $M_1(G_1 \times_{T^{110}} G_2) = n_1n_2(n_1-1)^2 + n_1M_1(G_2) + 4n_1m_2(n_1-1) + n_2m_1(m_1-1)^2$

(7) $M_1(G_1 \times_{T^{+10}} G_2) = n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2 + n_2m_1(m_1-1)^2$

(8) $M_1(G_1 \times_{T^{-10}} G_2) = n_1n_2(n_1-1)^2 + n_2M_1(G_1) - 4n_2m_1(n_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1-1) - 8m_1m_2 + n_2m_1(m_1-1)^2$

(9) $M_1(G_1 \times_{T^{0+0}} G_2) = n_1M_1(G_2) + n_2[HM(G_1) + 4m_1 - 4M_1(G_1)]$

(10) $M_1(G_1 \times_{T^{1+0}} G_2) = n_1n_2(n_1-1)^2 + n_1M_1(G_2) + 4n_1m_2(n_1-1) + n_2[HM(G_1) + 4m_1 - 4M_1(G_1)]$

(11) $M_1(G_1 \times_{T^{++0}} G_2) = n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2 + n_2[HM(G_1) + 4m_1 - 4M_1(G_1)]$

(12) $M_1(G_1 \times_{T^{-+0}} G_2) = n_1n_2(n_1-1)^2 + n_2M_1(G_1) - 4n_2m_1(n_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1-1) - 8m_1m_2 + n_2[HM(G_1) + 4m_1 - 4M_1(G_1)]$

(13) $M_1(G_1 \times_{T^{0-0}} G_2) = n_1M_1(G_2) + n_2[m_1(m_1+1)^2 + HM(G_1) - 2(m_1+1)M_1(G_1)]$

(14) $M_1(G_1 \times_{T^{1-0}} G_2) = n_1n_2(n_1-1)^2 + n_1M_1(G_2) + 4n_1m_2(n_1-1) + n_2[m_1(m_1+1)^2 + HM(G_1) - 2(m_1+1)M_1(G_1)]$

(15) $M_1(G_1 \times_{T^{+-0}} G_2) = n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2 + n_2[m_1(m_1+1)^2 + HM(G_1) - 2(m_1+1)M_1(G_1)]$

(16) $M_1(G_1 \times_{T^{--0}} G_2) = n_1n_2(n_1-1)^2 + n_2M_1(G_1) - 4n_2m_1(n_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1-1) - 8m_1m_2 + n_2[m_1(m_1+1)^2 + HM(G_1) - 2(m_1+1)M_1(G_1)]$.

Theorem 8. *Let G_1 and G_2 be the graphs. Then*

(1) $F(G_1 \times_{T^{000}} G_2) = n_1 F(G_2)$

(2) $F(G_1 \times_{T^{100}} G_2) = n_1 n_2 (n_1-1)^3 + n_1 F(G_2) + 6(n_1-1)^2 n_1 m_2 + 3(n_1-1) n_1 M_1(G_2)$

(3) $F(G_1 \times_{T^{+00}} G_2) = n_2 F(G_1) + n_1 F(G_2) + 6 m_2 M_1(G_1) + 6 m_1 M_1(G_2)$

(4) $F(G_1 \times_{T^{-00}} G_2) = n_1 n_2 (n_1-1)^3 - n_2 F(G_1) - 6 n_2 m_1 (n_1-1)^2 + 3 n_2 (n_1-1) M_1(G_1) + n_1 F(G_2) + 6 n_1 m_2 (n_1-1)^2 + 6 m_2 M_1(G_1) - 24 m_1 m_2 (n_1-1) + 3 n_1 (n_1-1) M_1(G_2) - 6 m_1 M_1(G_2)$

(5) $F(G_1 \times_{T^{010}} G_2) = n_1 F(G_2) + n_2 m_1 (m_1-1)^3$

(6) $F(G_1 \times_{T^{110}} G_2) = n_1 n_2 (n_1-1)^3 + n_1 F(G_2) + 6(n_1-1)^2 n_1 m_2 + 3(n_1-1) n_1 M_1(G_2) + n_2 m_1 (m_1-1)^3$

(7) $F(G_1 \times_{T^{+10}} G_2) = n_2 F(G_1) + n_1 F(G_2) + 6 m_2 M_1(G_1) + 6 m_1 M_1(G_2) + n_2 m_1 (m_1-1)^3$

(8) $F(G_1 \times_{T^{-10}} G_2) = n_1 n_2 (n_1-1)^3 - n_2 F(G_1) - 6 n_2 m_1 (n_1-1)^2 + 3 n_2 (n_1-1) M_1(G_1) + n_1 F(G_2) + 6 n_1 m_2 (n_1-1)^2 + 6 m_2 M_1(G_1) - 24 m_1 m_2 (n_1-1) + 3 n_1 (n_1-1) M_1(G_2) - 6 m_1 M_1(G_2) + n_2 m_1 (m_1-1)^3$

(9) $F(G_1 \times_{T^{0+0}} G_2) = n_1 F(G_2) + n_2 [M_1^3(G_1) - 8 m_1 - 6 HM(G_1) + 12 M_1(G_1)]$

(10) $F(G_1 \times_{T^{1+0}} G_2) = n_1 n_2 (n_1-1)^3 + n_1 F(G_2) + 6(n_1-1)^2 n_1 m_2 + 3(n_1-1) n_1 M_1(G_2) + n_2 [M_1^3(G_1) - 8 m_1 - 6 HM(G_1) + 12 M_1(G_1)]$

(11) $F(G_1 \times_{T^{++0}} G_2) = n_2 F(G_1) + n_1 F(G_2) + 6 m_2 M_1(G_1) + 6 m_1 M_1(G_2) + n_2 [M_1^3(G_1) - 8 m_1 - 6 HM(G_1) + 12 M_1(G_1)]$

(12) $F(G_1 \times_{T^{-+0}} G_2) = n_1 n_2 (n_1-1)^3 - n_2 F(G_1) - 6 n_2 m_1 (n_1-1)^2 + 3 n_2 (n_1-1) M_1(G_1) + n_1 F(G_2) + 6 n_1 m_2 (n_1-1)^2 + 6 m_2 M_1(G_1) - 24 m_1 m_2 (n_1-1) + 3 n_1 (n_1-1) M_1(G_2) - 6 m_1 M_1(G_2) + n_2 [M_1^3(G_1) - 8 m_1 - 6 HM(G_1) + 12 M_1(G_1)]$

(13) $F(G_1 \times_{T^{0-0}} G_2) = n_1 F(G_2) + n_2 [m_1 (m_1+1)^3 - M_1^3(G_1) - 3(m_1+1)^2 M_1(G_1) + 3(m_1+1) HM(G_1)]$

(14) $F(G_1 \times_{T^{1-0}} G_2) = n_1 n_2 (n_1-1)^3 + n_1 F(G_2) + 6(n_1-1)^2 n_1 m_2 + 3(n_1-1) n_1 M_1(G_2) + n_2 [m_1 (m_1+1)^3 - M_1^3(G_1) - 3(m_1+1)^2 M_1(G_1) + 3(m_1+1) HM(G_1)]$

(15) $F(G_1 \times_{T^{+-0}} G_2) = n_2 F(G_1) + n_1 F(G_2) + 6 m_2 M_1(G_1) + 6 m_1 M_1(G_2) + n_2 [m_1 (m_1+1)^3 - M_1^3(G_1) - 3(m_1+1)^2 M_1(G_1) + 3(m_1+1) HM(G_1)]$

(16) $F(G_1 \times_{T^{--0}} G_2) = n_1 n_2 (n_1-1)^3 - n_2 F(G_1) - 6 n_2 m_1 (n_1-1)^2 + 3 n_2 (n_1-1) M_1(G_1) + n_1 F(G_2) + 6 n_1 m_2 (n_1-1)^2 + 6 m_2 M_1(G_1) - 24 m_1 m_2 (n_1-1) + 3 n_1 (n_1-1) M_1(G_2) - 6 m_1 M_1(G_2) + n_2 [m_1 (m_1+1)^3 - M_1^3(G_1) - 3(m_1+1)^2 M_1(G_1) + 3(m_1+1) HM(G_1)]$.

By plugging the corresponding degrees of vertices of T^{xy1} from Proposition 2 in (3) and (4), we get the following two theorems.

Theorem 9. *Let G_1 and G_2 be the graphs. Then*

(1) $M_1(G_1 \times_{T^{001}} G_2) = n_1 n_2 m_1^2 + n_1 M_1(G_2) + 4 n_1 m_1 m_2 + n_2 m_1 n_1^2$

(2) $M_1(G_1 \times_{T^{101}} G_2) = n_1 n_2 (n_1 + m_1 - 1)^2 + n_1 M_1(G_2) + 4 n_1 m_2 (n_1 + m_1 - 1) + n_2 m_1 n_1^2$

(3) $M_1(G_1 \times_{T^{+01}} G_2) = n_1 n_2 m_1^2 + n_2 M_1(G_1) + 4 m_1^2 n_2 + n_1 M_1(G_2) + 4 m_1 m_2 n_1 + 8 m_1 m_2 + n_2 m_1 n_1^2$

(4) $M_1(G_1 \times_{T^{-01}} G_2) = n_1 n_2 (n_1 + m_1 - 1)^2 + n_2 M_1(G_1) - 4 m_1 n_2 (n_1 + m_1 - 1) + n_1 M_1(G_2) + 4 n_1 m_2 (n_1 + m_1 - 1) - 8 m_1 m_2 + n_2 m_1 n_1^2$

(5) $M_1(G_1 \times_{T^{011}} G_2) = n_1n_2m_1^2 + n_1M_1(G_2) + 4n_1m_1m_2 + n_2m_1(n_1+m_1-1)^2$

(6) $M_1(G_1 \times_{T^{111}} G_2) = n_1n_2(n_1+m_1-1)^2 + n_1M_1(G_2) + 4n_1m_2(n_1+m_1-1) + n_2m_1(n_1+m_1-1)^2$

(7) $M_1(G_1 \times_{T^{+11}} G_2) = n_1n_2m_1^2 + n_2M_1(G_1) + 4m_1^2n_2 + n_1M_1(G_2) + 4m_1m_2n_1 + 8m_1m_2 + n_2m_1(n_1+m_1-1)^2$

(8) $M_1(G_1 \times_{T^{-11}} G_2) = n_1n_2(n_1+m_1-1)^2 + n_2M_1(G_1) - 4m_1n_2(n_1+m_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1+m_1-1) - 8m_1m_2 + n_2m_1(n_1+m_1-1)^2$

(9) $M_1(G_1 \times_{T^{0+1}} G_2) = n_1n_2m_1^2 + n_1M_1(G_2) + 4n_1m_1m_2 + n_2[m_1(n_1-2)^2 + HM(G_1) + 2(n_1-2)M_1(G_1)]$

(10) $M_1(G_1 \times_{T^{1+1}} G_2) = n_1n_2(n_1+m_1-1)^2 + n_1M_1(G_2) + 4n_1m_2(n_1+m_1-1) + n_2[m_1(n_1-2)^2 + HM(G_1) + 2(n_1-2)M_1(G_1)]$

(11) $M_1(G_1 \times_{T^{++1}} G_2) = n_1n_2m_1^2 + n_2M_1(G_1) + 4m_1^2n_2 + n_1M_1(G_2) + 4m_1m_2n_1 + 8m_1m_2 + n_2[m_1(n_1-2)^2 + HM(G_1) + 2(n_1-2)M_1(G_1)]$

(12) $M_1(G_1 \times_{T^{-+1}} G_2) = n_1n_2(n_1+m_1-1)^2 + n_2M_1(G_1) - 4m_1n_2(n_1+m_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1+m_1-1) - 8m_1m_2 + n_2[m_1(n_1-2)^2 + HM(G_1) + 2(n_1-2)M_1(G_1)]$

(13) $M_1(G_1 \times_{T^{0-1}} G_2) = n_1n_2m_1^2 + n_1M_1(G_2) + 4n_1m_1m_2 + n_2[m_1(n_1+m_1+1)^2 + HM(G_1) - 2(n_1+m_1+1)M_1(G_1)]$

(14) $M_1(G_1 \times_{T^{1-1}} G_2) = n_1n_2(n_1+m_1+1)^2 + n_1M_1(G_2) + 4n_1m_2(n_1+m_1-1) + n_2[m_1(n_1+m_1+1)^2 + HM(G_1) - 2(n_1+m_1+1)M_1(G_1)]$

(15) $M_1(G_1 \times_{T^{+-1}} G_2) = n_1n_2m_1^2 + n_2M_1(G_1) + 4m_1^2n_2 + n_1M_1(G_2) + 4m_1m_2n_1 + 8m_1m_2 + n_2[m_1(n_1+m_1+1)^2 + HM(G_1) - 2(n_1+m_1+1)M_1(G_1)]$

(16) $M_1(G_1 \times_{T^{--1}} G_2) = n_1n_2(n_1+m_1-1)^2 + n_2M_1(G_1) - 4m_1n_2(n_1+m_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1+m_1-1) - 8m_1m_2 + n_2[m_1(n_1+m_1+1)^2 + HM(G_1) - 2(n_1+m_1+1)M_1(G_1)]$.

Theorem 10. *Let G_1 and G_2 be the graphs. Then*

(1) $F(G_1 \times_{T^{001}} G_2) = n_1n_2m_1^3 + n_1F(G_2) + 6n_1m_2m_1^2 + 3m_1n_1M_1(G_2) + n_2m_1n_1^3$

(2) $F(G_1 \times_{T^{101}} G_2) = n_1n_2(n_1+m_1-1)^3 + n_1F(G_2) + 6n_1m_2(n_1+m_1-1)^2 + 3n_1(n_1+m_1-1)M_1(G_2) + n_2m_1n_1^3$

(3) $F(G_1 \times_{T^{+01}} G_2) = n_1n_2m_1^3 + n_2F(G_1) + 6n_2m_1^3 + 3m_1n_2M_1(G_1) + n_1F(G_2) + 6m_1^2n_1m_2 + 6m_2M_1(G_1) + 24m_1^2m_2 + n_2m_1n_1^3$

(4) $F(G_1 \times_{T^{-01}} G_2) = n_1n_2(n_1+m_1-1)^3 - n_2F(G_1) + (6n_1m_2 - 6n_2m_1)(n_1+m_1-1)^2 + (3n_2M_1(G_1) + 3n_1M_1(G_2) - 24m_1m_2)(n_1+m_1-1) + n_1F(G_2) + 6m_2M_1(G_1) - 6m_1M_1(G_2) + n_2m_1n_1^3$

(5) $F(G_1 \times_{T^{011}} G_2) = n_1n_2m_1^3 + n_1F(G_2) + 6n_1m_2m_1^2 + 3m_1n_1M_1(G_2) + n_2m_1(n_1+m_1-1)^3$

(6) $F(G_1 \times_{T^{111}} G_2) = n_1n_2(n_1+m_1-1)^3 + n_1F(G_2) + 6n_1m_2(n_1+m_1-1)^2 + 3n_1(n_1+m_1-1)M_1(G_2) + n_2m_1(n_1+m_1-1)^3$

(7) $F(G_1 \times_{T^{+11}} G_2) = n_1n_2m_1^3 + n_2F(G_1) + 6n_2m_1^3 + 3m_1n_2M_1(G_1) + n_1F(G_2) + 6m_1^2n_1m_2 + 6m_2M_1(G_1) + 24m_1^2m_2 + n_2m_1(n_1+m_1-1)^3$

(8) $F(G_1 \times_{T^{-11}} G_2) = n_1n_2(n_1+m_1-1)^3 - n_2F(G_1) + (6n_1m_2 - 6n_2m_1)(n_1+m_1-1)^2 + (3n_2M_1(G_1) + 3n_1M_1(G_2) - 24m_1m_2)(n_1+m_1-1) + n_1F(G_2) + 6m_2M_1(G_1) - 6m_1M_1(G_2) + n_2m_1(n_1+m_1-1)^3$

(9) $F(G_1 \times_{T^{0+1}} G_2) = n_1n_2m_1^3 + n_1F(G_2) + 6n_1m_2m_1^2 + 3m_1n_1M_1(G_2) + n_2[m_1(n_1-2)^3 + M_1^3(G_1) + 3(n_1-2)^2M_1(G_1) + 3(n_1-2)HM(G_1)]$

(10) $F(G_1 \times_{T^{1+1}} G_2) = n_1n_2(n_1+m_1-1)^3 + n_1F(G_2) + 6n_1m_2(n_1+m_1-1)^2 + 3n_1(n_1+m_1-1)M_1(G_2) + n_2[m_1(n_1-2)^3 + M_1^3(G_1) + 3(n_1-2)^2M_1(G_1) + 3(n_1-2)HM(G_1)]$

(11) $F(G_1 \times_{T^{++1}} G_2) = n_1n_2m_1^3 + n_2F(G_1) + 6n_2m_1^3 + 3m_1n_2M_1(G_1) + n_1F(G_2) + 6m_1^2n_1m_2 + 6m_2M_1(G_1) + 24m_1^2m_2 + n_2[m_1(n_1-2)^3 + M_1^3(G_1) + 3(n_1-2)^2M_1(G_1) + 3(n_1-2)HM(G_1)]$

(12) $F(G_1 \times_{T^{-+1}} G_2) = n_1n_2(n_1+m_1-1)^3 - n_2F(G_1) + (6n_1m_2 - 6n_2m_1)(n_1+m_1-1)^2 + (3n_2M_1(G_1) + 3n_1M_1(G_2) - 24m_1m_2)(n_1+m_1-1) + n_1F(G_2) + 6m_2M_1(G_1) - 6m_1M_1(G_2) + n_2[m_1(n_1-2)^3 + M_1^3(G_1) + 3(n_1-2)^2M_1(G_1) + 3(n_1-2)HM(G_1)]$

(13) $F(G_1 \times_{T^{0-1}} G_2) = n_1n_2m_1^3 + n_1F(G_2) + 6n_1m_2m_1^2 + 3m_1n_1M_1(G_2) + n_2[m_1(n_1+m_1+1)^3 - M_1^3(G_1) - 3(n_1+m_1+1)^2M_1(G_1) + 3(n_1+m_1+1)HM(G_1)]$

(14) $F(G_1 \times_{T^{1-1}} G_2) = n_1n_2(n_1+m_1-1)^3 + n_1F(G_2) + 6n_1m_2(n_1+m_1-1)^2 + 3n_1(n_1+m_1-1)M_1(G_2) + n_2[m_1(n_1+m_1+1)^3 - M_1^3(G_1) - 3(n_1+m_1+1)^2M_1(G_1) + 3(n_1+m_1+1)HM(G_1)]$

(15) $F(G_1 \times_{T^{+-1}} G_2) = n_1n_2m_1^3 + n_2F(G_1) + 6n_2m_1^3 + 3m_1n_2M_1(G_1) + n_1F(G_2) + 6m_1^2n_1m_2 + 6m_2M_1(G_1) + 24m_1^2m_2 + n_2[m_1(n_1+m_1+1)^3 - M_1^3(G_1) - 3(n_1+m_1+1)^2M_1(G_1) + 3(n_1+m_1+1)HM(G_1)]$

(16) $F(G_1 \times_{T^{--1}} G_2) = n_1n_2(n_1+m_1-1)^3 - n_2F(G_1) + (6n_1m_2 - 6n_2m_1)(n_1+m_1-1)^2 + (3n_2M_1(G_1) + 3n_1M_1(G_2) - 24m_1m_2)(n_1+m_1-1) + n_1F(G_2) + 6m_2M_1(G_1) - 6m_1M_1(G_2) + n_2[m_1(n_1+m_1+1)^3 - M_1^3(G_1) - 3(n_1+m_1+1)^2M_1(G_1) + 3(n_1+m_1+1)HM(G_1)]$.

By plugging the corresponding degrees of vertices of T^{xy+} from Proposition 3 in (3) and (4), we get the following two theorems.

Theorem 11. *Let G_1 and G_2 be the graphs. Then*

(1) $M_1(G_1 \times_{T^{00+}} G_2) = n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2 + 4n_2m_1$

(2) $M_1(G_1 \times_{T^{10+}} G_2) = n_1n_2(n_1-1)^2 + n_2M_1(G_1) + 4n_2m_1(n_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1-1) + 8m_1m_2 + 4n_2m_1$

(3) $M_1(G_1 \times_{T^{+0+}} G_2) = 4n_2M_1(G_1) + n_1M_1(G_2) + 16m_1m_2 + 4n_2m_1$

(4) $M_1(G_1 \times_{T^{-0+}} G_2) = n_1n_2(n_1-1)^2 + n_1M_1(G_2) + 4n_1m_2(n_1-1) + 4n_2m_1$

(5) $M_1(G_1 \times_{T^{01+}} G_2) = n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2 + n_2m_1(m_1+1)^2$

(6) $M_1(G_1 \times_{T^{11+}} G_2) = n_1n_2(n_1-1)^2 + n_2M_1(G_1) + 4n_2m_1(n_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1-1) + 8m_1m_2 + n_2m_1(m_1+1)^2$

(7) $M_1(G_1 \times_{T^{+1+}} G_2) = 4n_2M_1(G_1) + n_1M_1(G_2) + 16m_1m_2 + n_2m_1(m_1+1)^2$

(8) $M_1(G_1 \times_{T^{-1+}} G_2) = n_1n_2(n_1-1)^2 + n_1M_1(G_2) + 4n_1m_2(n_1-1) + n_2m_1(m_1+1)^2$

(9) $M_1(G_1 \times_{T^{0++}} G_2) = n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2 + n_2HM(G_1)$

(10) $M_1(G_1 \times_{T^{1++}} G_2) = n_1n_2(n_1-1)^2 + n_2M_1(G_1) + 4n_2m_1(n_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1-1) + 8m_1m_2 + n_2HM(G_1)$

(11) $M_1(G_1 \times_{T^{+++}} G_2) = 4n_2M_1(G_1) + n_1M_1(G_2) + 16m_1m_2 + n_2HM(G_1)$

(12) $M_1(G_1 \times_{T^{-++}} G_2) = n_1n_2(n_1-1)^2 + n_1M_1(G_2) + 4n_1m_2(n_1-1) + n_2HM(G_1)$

(13) $M_1(G_1 \times_{T^{0-+}} G_2) = n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2 + n_2[m_1(m_1+3)^2 + HM(G_1) - 2(m_1+3)M_1(G_1)]$

(14) $M_1(G_1 \times_{T^{1-+}} G_2) = n_1n_2(n_1-1)^2 + n_2M_1(G_1) + 4n_2m_1(n_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1-1) + 8m_1m_2 + n_2[m_1(m_1+3)^2 + HM(G_1) - 2(m_1+3)M_1(G_1)]$

(15) $M_1(G_1 \times_{T^{+-+}} G_2) = 4n_2M_1(G_1) + n_1M_1(G_2) + 16m_1m_2 + n_2[m_1(m_1+3)^2 + HM(G_1) - 2(m_1+3)M_1(G_1)]$

(16) $M_1(G_1 \times_{T^{--+}} G_2) = n_1n_2(n_1-1)^2 + n_1M_1(G_2) + 4n_1m_2(n_1-1) + n_2[m_1(m_1+3)^2 + HM(G_1) - 2(m_1+3)M_1(G_1)]$.

Theorem 12. *Let G_1 and G_2 be the graphs. Then*

(1) $F(G_1 \times_{T^{00+}} G_2) = n_2F(G_1) + n_1F(G_2) + 6m_2M_1(G_1) + 6m_1M_1(G_2) + 8n_2m_1$

(2) $F(G_1 \times_{T^{10+}} G_2) = n_1n_2(n_1-1)^3 + n_2F(G_1) + 6(n_1-1)^2n_2m_1 + 3(n_1-1)n_2M_1(G_1) + n_1F(G_2) + 6n_1m_2(n_1-1)^2 + 6m_2M_1(G_1) + 24m_1m_2(n_1-1) + 3n_1(n_1-1)M_1(G_2) + 6m_1M_1(G_2) + 8n_2m_1$

(3) $F(G_1 \times_{T^{+0+}} G_2) = 8n_2F(G_1) + n_1F(G_2) + 24m_2M_1(G_1) + 12m_1M_1(G_2) + 8n_2m_1$

(4) $F(G_1 \times_{T^{-0+}} G_2) = n_1n_2(n_1-1)^3 + n_1F(G_2) + 6n_1m_2(n_1-1)^2 + 3(n_1-1)n_1M_1(G_2) + 8n_2m_1$

(5) $F(G_1 \times_{T^{01+}} G_2) = n_2F(G_1) + n_1F(G_2) + 6m_2M_1(G_1) + 6m_1M_1(G_2) + n_2m_1(m_1+1)^3$

(6) $F(G_1 \times_{T^{11+}} G_2) = n_1n_2(n_1-1)^3 + n_2F(G_1) + 6(n_1-1)^2n_2m_1 + 3(n_1-1)n_2M_1(G_1) + n_1F(G_2) + 6n_1m_2(n_1-1)^2 + 6m_2M_1(G_1) + 24m_1m_2(n_1-1) + 3n_1(n_1-1)M_1(G_2) + 6m_1M_1(G_2) + n_2m_1(m_1+1)^3$

(7) $F(G_1 \times_{T^{+1+}} G_2) = 8n_2F(G_1) + n_1F(G_2) + 24m_2M_1(G_1) + 12m_1M_1(G_2) + n_2m_1(m_1+1)^3$

(8) $F(G_1 \times_{T^{-1+}} G_2) = n_1n_2(n_1-1)^3 + n_1F(G_2) + 6n_1m_2(n_1-1)^2 + 3(n_1-1)n_1M_1(G_2) + n_2m_1(m_1+1)^3$

(9) $F(G_1 \times_{T^{0++}} G_2) = n_2F(G_1) + n_1F(G_2) + 6m_2M_1(G_1) + 6m_1M_1(G_2) + n_2M_1^3(G_1)$

(10) $F(G_1 \times_{T^{1++}} G_2) = n_1n_2(n_1-1)^3 + n_2F(G_1) + 6(n_1-1)^2n_2m_1 + 3(n_1-1)n_2M_1(G_1) + n_1F(G_2) + 6n_1m_2(n_1-1)^2 + 6m_2M_1(G_1) + 24m_1m_2(n_1-1) + 3n_1(n_1-1)M_1(G_2) + 6m_1M_1(G_2) + n_2M_1^3(G_1)$

(11) $F(G_1 \times_{T^{+++}} G_2) = 8n_2F(G_1) + n_1F(G_2) + 24m_2M_1(G_1) + 12m_1M_1(G_2) + n_2M_1^3(G_1)$

(12) $F(G_1 \times_{T^{-++}} G_2) = n_1n_2(n_1-1)^3 + n_1F(G_2) + 6n_1m_2(n_1-1)^2 + 3(n_1-1)n_1M_1(G_2) + n_2M_1^3(G_1)$

(13) $F(G_1 \times_{T^{0-+}} G_2) = n_2F(G_1) + n_1F(G_2) + 6m_2M_1(G_1) + 6m_1M_1(G_2) + n_2[m_1(m_1+3)^3 - M_1^3(G_1) - 3(m_1+3)^2M_1(G_1) + 3(m_1+3)HM(G_1)]$

(14) $F(G_1 \times_{T^{1-+}} G_2) = n_1n_2(n_1-1)^3 + n_2F(G_1) + 6(n_1-1)^2n_2m_1 + 3(n_1-1)n_2M_1(G_1) + n_1F(G_2) + 6n_1m_2(n_1-1)^2 + 6m_2M_1(G_1) + 24m_1m_2(n_1-1) + 3n_1(n_1-1)M_1(G_2) + 6m_1M_1(G_2) + n_2[m_1(m_1+3)^3 - M_1^3(G_1) - 3(m_1+3)^2M_1(G_1) + 3(m_1+3)HM(G_1)]$

(15) $F(G_1 \times_{T^{+-+}} G_2) = 8n_2F(G_1) + n_1F(G_2) + 24m_2M_1(G_1) + 12m_1M_1(G_2) + n_2[m_1(m_1+3)^3 - M_1^3(G_1) - 3(m_1+3)^2M_1(G_1) + 3(m_1+3)HM(G_1)]$

(16) $F(G_1 \times_{T^{--+}} G_2) = n_1n_2(n_1-1)^3 + n_1F(G_2) + 6n_1m_2(n_1-1)^2 + 3(n_1-1)n_1M_1(G_2) + n_2[m_1(m_1+3)^3 - M_1^3(G_1) - 3(m_1+3)^2M_1(G_1) + 3(m_1+3)HM(G_1)]$.

By plugging the corresponding degrees of vertices of T^{xy-} from Proposition 4 in (3) and (4), we reach the following two theorems.

Theorem 13. *Let G_1 and G_2 be the graphs. Then*

(1) $M_1(G_1 \times_{T^{00-}} G_2) = n_1n_2m_1^2 + n_2M_1(G_1) - 4m_1^2n_2 + n_1M_1(G_2) + 4m_1m_2n_1 - 8m_1m_2 + n_2m_1(n_1-2)^2$

(2) $M_1(G_1 \times_{T^{10-}} G_2) = n_1n_2(n_1+m_1-1)^2 + n_2M_1(G_1) - 4m_1n_2(n_1+m_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1+m_1-1) - 8m_1m_2 + n_2m_1(n_1-2)^2$

(3) $M_1(G_1 \times_{T^{+0-}} G_2) = n_1n_2m_1^2 + n_1M_1(G_2) + 4m_1m_2n_1 + n_2m_1(n_1-2)^2$

(4) $M_1(G_1 \times_{T^{-0-}} G_2) = n_1n_2(n_1+m_1-1)^2 + 4n_2M_1(G_1) - 8m_1n_2(n_1+m_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1+m_1-1) - 16m_1m_2 + n_2m_1(n_1-2)^2$

(5) $M_1(G_1 \times_{T^{01-}} G_2) = n_1n_2m_1^2 + n_2M_1(G_1) - 4m_1^2n_2 + n_1M_1(G_2) + 4m_1m_2n_1 - 8m_1m_2 + n_2m_1(n_1+m_1-3)^2$

(6) $M_1(G_1 \times_{T^{11-}} G_2) = n_1n_2(n_1+m_1-1)^2 + n_2M_1(G_1) - 4m_1n_2(n_1+m_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1+m_1-1) - 8m_1m_2 + n_2m_1(n_1+m_1-3)^2$

(7) $M_1(G_1 \times_{T^{+1-}} G_2) = n_1n_2m_1^2 + n_1M_1(G_2) + 4m_1m_2n_1 + n_2m_1(n_1+m_1-3)^2$

(8) $M_1(G_1 \times_{T^{-1-}} G_2) = n_1n_2(n_1+m_1-1)^2 + 4n_2M_1(G_1) - 8m_1n_2(n_1+m_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1+m_1-1) - 16m_1m_2 + n_2m_1(n_1+m_1-3)^2$

(9) $M_1(G_1 \times_{T^{0+-}} G_2) = n_1n_2m_1^2 + n_2M_1(G_1) - 4m_1^2n_2 + n_1M_1(G_2) + 4m_1m_2n_1 - 8m_1m_2 + n_2[m_1(n_1-4)^2 + HM(G_1) + 2(n_1-4)M_1(G_1)]$

(10) $M_1(G_1 \times_{T^{1+-}} G_2) = n_1n_2(n_1+m_1-1)^2 + n_2M_1(G_1) - 4m_1n_2(n_1+m_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1+m_1-1) - 8m_1m_2 + n_2[m_1(n_1-4)^2 + HM(G_1) + 2(n_1-4)M_1(G_1)]$

(11) $M_1(G_1 \times_{T^{++-}} G_2) = n_1n_2m_1^2 + n_1M_1(G_2) + 4m_1m_2n_1 + n_2[m_1(n_1-4)^2 + HM(G_1) + 2(n_1-4)M_1(G_1)]$

(12) $M_1(G_1 \times_{T^{-+-}} G_2) = n_1n_2(n_1+m_1-1)^2 + 4n_2M_1(G_1) - 8m_1n_2(n_1+m_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1+m_1-1) - 16m_1m_2 + n_2[m_1(n_1-4)^2 + HM(G_1) + 2(n_1-4)M_1(G_1)]$

(13) $M_1(G_1 \times_{T^{0--}} G_2) = n_1n_2m_1^2 + n_2M_1(G_1) - 4m_1^2n_2 + n_1M_1(G_2) + 4m_1m_2n_1 - 8m_1m_2 + n_2[m_1(n_1+m_1-1)^2 + HM(G_1) - 2(n_1+m_1-1)M_1(G_1)]$

(14) $M_1(G_1 \times_{T^{1--}} G_2) = n_1n_2(n_1+m_1-1)^2 + n_2M_1(G_1) - 4m_1n_2(n_1+m_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1+m_1-1) - 8m_1m_2 + n_2[m_1(n_1+m_1-1)^2 + HM(G_1) - 2(n_1+m_1-1)M_1(G_1)]$

(15) $M_1(G_1 \times_{T^{+--}} G_2) = n_1n_2m_1^2 + n_1M_1(G_2) + 4m_1m_2n_1 + n_2[m_1(n_1+m_1-1)^2 + HM(G_1) - 2(n_1+m_1-1)M_1(G_1)]$

(16) $M_1(G_1 \times_{T^{---}} G_2) = n_1n_2(n_1+m_1-1)^2 + 4n_2M_1(G_1) - 8m_1n_2(n_1+m_1-1) + n_1M_1(G_2) + 4n_1m_2(n_1+m_1-1) - 16m_1m_2 + n_2[m_1(n_1+m_1-1)^2 + HM(G_1) - 2(n_1+m_1-1)M_1(G_1)]$.

Theorem 14. *Let G_1 and G_2 be the graphs. Then*

(1) $F(G_1 \times_{T^{00-}} G_2) = n_1n_2m_1^3 - n_2F(G_1) - 6m_1^3n_2 + 3m_1n_2M_1(G_1) + n_1F(G_2) + 6m_1^2n_1m_2 + 6m_2M_1(G_1) - 24m_1^2m_2 + 3m_1n_1M_1(G_2) - 6m_1M_1(G_2) + n_2m_1(n_1-2)^3$

(2) $F(G_1 \times_{T^{10-}} G_2) = n_1n_2(n_1+m_1-1)^3 - n_2F(G_1) + (6n_1m_2 - 6m_1n_2)(n_1+m_1-1)^2 + (3n_2M_1(G_1) + 3n_1M_1(G_2) - 24m_1m_2)(n_1+m_1-1) + n_1F(G_2) - 6m_1M_1(G_2) + 6m_2M_1(G_1) + n_2m_1(n_1-2)^3$

(3) $F(G_1 \times_{T^{+0-}} G_2) = n_1n_2m_1^3 + n_1F(G_2) + 6n_1m_2m_1^2 + 3m_1n_1M_1(G_2) + n_2m_1(n_1-2)^3$

(4) $F(G_1 \times_{T^{-0-}} G_2) = n_1n_2(n_1+m_1-1)^3 + (6n_1m_2 - 12n_2m_1)(n_1+m_1-1)^2 + (12n_2M_1(G_1) + 3n_1M_1(G_2) - 48m_1m_2)(n_1+m_1-1) - 8n_2F(G_1) + n_1F(G_2) + 24m_2M_1(G_1) - 12m_1M_1(G_2) + n_2m_1(n_1-2)^3$

(5) $F(G_1 \times_{T^{01-}} G_2) = n_1n_2m_1^3 - n_2F(G_1) - 6m_1^3n_2 + 3m_1n_2M_1(G_1) + n_1F(G_2) + 6m_1^2n_1m_2 + 6m_2M_1(G_1) - 24m_1^2m_2 + 3m_1n_1M_1(G_2) - 6m_1M_1(G_2) + n_2m_1(n_1+m_1-3)^3$

(6) $F(G_1 \times_{T^{11-}} G_2) = n_1n_2(n_1+m_1-1)^3 - n_2F(G_1) + (6n_1m_2 - 6m_1n_2)(n_1+m_1-1)^2 + (3n_2M_1(G_1) + 3n_1M_1(G_2) - 24m_1m_2)(n_1+m_1-1) + n_1F(G_2) - 6m_1M_1(G_2) + 6m_2M_1(G_1) + n_2m_1(n_1+m_1-3)^3$

(7) $F(G_1 \times_{T^{+1-}} G_2) = n_1n_2m_1^3 + n_1F(G_2) + 6n_1m_2m_1^2 + 3m_1n_1M_1(G_2) + n_2m_1(n_1+m_1-3)^3$

(8) $F(G_1 \times_{T^{-1-}} G_2) = n_1n_2(n_1+m_1-1)^3 + (6n_1m_2 - 12n_2m_1)(n_1+m_1-1)^2 + (12n_2M_1(G_1) + 3n_1M_1(G_2) - 48m_1m_2)(n_1+m_1-1) - 8n_2F(G_1) + n_1F(G_2) + 24m_2M_1(G_1) - 12m_1M_1(G_2) + n_2m_1(n_1+m_1-3)^3$

(9) $F(G_1 \times_{T^{0+-}} G_2) = n_1n_2m_1^3 - n_2F(G_1) - 6m_1^3n_2 + 3m_1n_2M_1(G_1) + n_1F(G_2) + 6m_1^2n_1m_2 + 6m_2M_1(G_1) - 24m_1^2m_2 + 3m_1n_1M_1(G_2) - 6m_1M_1(G_2) + n_2[m_1(n_1-4)^3 + M_1^3(G_1) + 3(n_1-4)^2M_1(G_1) + 3(n_1-4)HM(G_1)]$

(10) $F(G_1 \times_{T^{1+-}} G_2) = n_1n_2(n_1+m_1-1)^3 - n_2F(G_1) + (6n_1m_2 - 6m_1n_2)(n_1+m_1-1)^2 + (3n_2M_1(G_1) + 3n_1M_1(G_2) - 24m_1m_2)(n_1+m_1-1) + n_1F(G_2) - 6m_1M_1(G_2) + 6m_2M_1(G_1) + n_2[m_1(n_1-4)^3 + M_1^3(G_1) + 3(n_1-4)^2M_1(G_1) + 3(n_1-4)HM(G_1)]$

(11) $F(G_1 \times_{T^{++-}} G_2) = n_1n_2m_1^3 + n_1F(G_2) + 6n_1m_2m_1^2 + 3m_1n_1M_1(G_2) + n_2[m_1(n_1-4)^3 + M_1^3(G_1) + 3(n_1-4)^2M_1(G_1) + 3(n_1-4)HM(G_1)]$

(12) $F(G_1 \times_{T^{-+-}} G_2) = n_1n_2(n_1+m_1-1)^3 + (6n_1m_2 - 12n_2m_1)(n_1+m_1-1)^2 + (12n_2M_1(G_1) + 3n_1M_1(G_2) - 48m_1m_2)(n_1+m_1-1) - 8n_2F(G_1) + n_1F(G_2) + 24m_2M_1(G_1) - 12m_1M_1(G_2) + n_2[m_1(n_1-4)^3 + M_1^3(G_1) + 3(n_1-4)^2M_1(G_1) + 3(n_1-4)HM(G_1)]$

(13) $F(G_1 \times_{T^{0--}} G_2) = n_1n_2m_1^3 - n_2F(G_1) - 6m_1^3n_2 + 3m_1n_2M_1(G_1) + n_1F(G_2) + 6m_1^2n_1m_2 + 6m_2M_1(G_1) - 24m_1^2m_2 + 3m_1n_1M_1(G_2) - 6m_1M_1(G_2) + n_2[m_1(n_1+m_1-1)^3 - M_1^3(G_1) - 3(n_1+m_1-1)^2M_1(G_1) + 3(n_1+m_1-1)HM(G_1)]$

(14) $F(G_1 \times_{T^{1--}} G_2) = n_1n_2(n_1+m_1-1)^3 - n_2F(G_1) + (6n_1m_2 - 6m_1n_2)(n_1+m_1-1)^2 + (3n_2M_1(G_1) + 3n_1M_1(G_2) - 24m_1m_2)(n_1+m_1-1) + n_1F(G_2) - 6m_1M_1(G_2) + 6m_2M_1(G_1) + n_2[m_1(n_1+m_1-1)^3 - M_1^3(G_1) - 3(n_1+m_1-1)^2M_1(G_1) + 3(n_1+m_1-1)HM(G_1)]$

(15) $F(G_1 \times_{T^{+--}} G_2) = n_1n_2m_1^3 + n_1F(G_2) + 6n_1m_2m_1^2 + 3m_1n_1M_1(G_2) + n_2[m_1(n_1+m_1-1)^3 - M_1^3(G_1) - 3(n_1+m_1-1)^2M_1(G_1) + 3(n_1+m_1-1)HM(G_1)]$

(16) $F(G_1 \times_{T^{---}} G_2) = n_1n_2(n_1+m_1-1)^3 + (6n_1m_2 - 12n_2m_1)(n_1+m_1-1)^2 + (12n_2M_1(G_1) + 3n_1M_1(G_2) - 48m_1m_2)(n_1+m_1-1) - 8n_2F(G_1) + n_1F(G_2) + 24m_2M_1(G_1) - 12m_1M_1(G_2) + n_2[m_1(n_1+m_1-1)^3 - M_1^3(G_1) - 3(n_1+m_1-1)^2M_1(G_1) + 3(n_1+m_1-1)HM(G_1)]$.

The expression for first Zagreb index of eighteen graph operations $G_1 \times_{T^{xyz}} G_2$ for $x,y,z \in \{+,-\}$, $x,z \in \{+,-\}$ with $y = 0$, $y,z \in \{+,-\}$ with $x = 0$ and $z \in \{+,-\}$ with $x = y = 0$ are obtained by Basavanagoud and Patil in [6]. We include these results for the sake of completeness.

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References

[1] M. An, L. Xiong, K. C. Das, Two upper bounds for the degree distances of four sums of graphs, *Filomat*, **28(3)** (2014) 579–590.

[2] B. Basavanagoud, I. Gutman, V. R. Desai, Zagreb indices of generalized transformation graphs and their complements, *Kragujevac J. Sci.* **37** (2015) 99-112.
[3] B. Basavanagoud, S. Patil, A note on hyper-Zagreb index of graph operations, *Iranian J. Math. Chem.* **7**(1), (2016) 89–92.
[4] B. Basavanagoud, S. Patil, A note on hyper-Zagreb coindex of graph operations, *J. Appl. Math. Comput.*, **53** (2017) 647–655.
[5] B. Basavanagoud, S. Patil, The hyper-Zagreb index of four operations on graphs, Math. Sci. Lett., **6(2)** (2017) 193–198.
[6] B. Basavanagoud, S. Patil, Generalized four new sums of graphs and their Zagreb indices, communicated.
[7] A. Deng, A. Kelmans, J. Meng, Laplacian spectra of regular graph transformations, *Discrete Appl. Math.*, **161** (2013) 118–133.
[8] H. Deng, D. Sarala, S. K. Ayyaswamy, S. Balachandran, The Zagreb indices of four operations on graphs, *Appl. Math. Comput.* **275** (2016) 422–431.
[9] M. Eliasi, B. Taer, Four new sums of graphs and their Wiener indices, *Discrete Appl. Math.*, **157** (2009) 794–803.
[10] F. Furtula, I. Gutman, A forgotten topological index, *J. Math. Chem.*, **53** (2015) 1184–1190.
[11] W. Gao, L. Yan, L. Shi, Generalized Zagreb index of polyomino chains and nanotubes, *Optoelectronics and Advanced Materials-Rapid Communications*, **11**(1-2) (2017) 119–124.
[12] W. Gao, M. K. Siddiqui, Molecular descriptors of nanotube, oxide, silicate, and triangulene networks, *J. Chem.*, Volume 2017, Article ID 6540754, 10 pages, https://doi.org/10.1155/2017/6540754.
[13] W. Gao, M. K. Jamil, A. Javed. M. R. Farahani, S. Wang, J. Liu, Sharp bounds of the hyper Zagreb index on acyclic, unicylic and bicyclic graphs, *Discrete Dynamics in Nature and Society*, Volume 2017, Article ID 6079450, 5 pages, https://doi.org/10.1155/2017/6079450.
[14] A. Graovac, T. Pisanski, On the Wiener index of a graph, *J. Math. Chem.*, **8** (1991) 53–62.
[15] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.*, **17** (1972) 535–538.
[16] I. Gutman, O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer, Berlin (1986).
[17] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass (1969).
[18] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, I. Gutman, The edge Szeged index of product graphs, *Croat. Chem. Acta*, **81** (2008) 277–281.
[19] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, Vertex and Edge PI indices of Cartesian product graphs, *Discrete Appl. Math.*, **156** (2008) 1780–1789.
[20] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, The first and second Zagreb indices of some graph operations, *Discrete Appl. Math.*, **157** (2009) 804–811.
[21] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, S. Wagner, Some new results on distance-based graph invariants, *European J. Combin.*, **30** (2009) 1149–1163.
[22] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, The hyper-Wiener index of graph operations, *Comput. Math. Appl.*, **56** (2008) 1402–1407.
[23] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, A matrix method for computing Szeged and vertex PI indices of join and composition of graphs, *Linear Algebra Appl.*, **429** (2008) 2702–2709.
[24] S. Klavzar, A. Rajapakse, I. Gutman, The Szeged and the Wiener index of graphs, *Appl. Math. Lett.*, **9** (1996) 45–49.
[25] V. R. Kulli, *College Graph Theory*, Vishwa International Publications, Gulbarga, India (2012).
[26] S. Li, G. Wang, Vertex PI indices of four sums of graphs, *Discrete Appl. Math.*, **159** (2011) 1601–1607.
[27] X. Li, H. Zhao, Trees with the first three smallest and largest generalized topological indices, *MATCH Commun. Math. Comput. Chem.*, **50** (2004) 57–62.
[28] M. Metsidik, W. Zhang, F. Duan, Hyper and reverse Wiener indices of F-sums of graphs, *Discrete Appl. Math.*, **158** (2010) 1433–1440.
[29] G. H. Shirdel, H. Rezapour, A. M. Sayadi, The hyper-Zagreb index of graph operations, *Iranian J. Math. Chem.*, **4(2)** (2013) 213–220.
[30] N. Trinajstić, *Chemical Graph Theory*, CRC Press, Boca Raton, FL (1992).
[31] B. Zhou, N. Trinajstić, On general sum-connectivity index, *J. Math. Chem.*, **47** (2010) 210–218.