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Weakly Axioms in Topological Spaces

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Abstract

The aim of this paper is to introduce and study two new classes of spaces, namely Weakly-normal and weakly-regular spaces and obtained their properties by utilizing weakly-closed sets.

Keywords: Weakly-closed set, Weakly-continuous function, Weakly-Separation axioms.
Mathematics subject classification (2010): 54A05.

1 Introduction

S.S. Benchalli, T.D. Rayanagoudar and P.G. Patil introduced the concept of g*-closed sets And S.S. Benchalli, T.D. Rayanagoudar and P.G. Patiland Shik John studied the concept of g*- preregular, g*- pre normal and obtained their properties by utilizing g*-closed sets.

2 Preliminaries

Throughout this paper space (X, τ) and (Y, σ) (or simply X and Y) always denote topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X, Cl(A), Int(A), A^c , and α-Cl(A), denote the Closure of A, Interior of A and Compliment of A and α-closure of A in X respectively.

Definition 2.1: A subset A of a topological space (X, τ) is called
(i)W-closed set[12] if cl(A) ⊆ U whenever A⊆ U and U is semi-open in X.
(ii)Generalized closed set(briefly g-closed) [7] if cl(A)⊆U whenever A ⊆ U and U is open in X.

Definition 2.2 :A topological space X is said to be a
(1)g-regular[10], if for each g-closed set F of X and each point x ∉F,there exists disjoint open sets U and V such that F⊆U and x ϵ V .
(2) α - regular [4], if for each α - closed set F of X and each point x ∉ F, there existsdisjoint α - open sets U and V such that F ⊆V and x ϵ U.
(3) w-regular[12], if for each closed set F of X and each point x ∉ F, there existsdisjoint w-open sets U and V such that F⊆U and xϵV.

Definition 2.3.A topological space X is said to be a
(1) g- normal [10], if for any pair of disjoint g-closed sets A and B, there exists disjoint open sets U and V such that A⊆U and B⊆V .
(2) α-normal [4], if for any pair of disjoint $\alpha -$ closed sets A and B, there exists dis-joint α-open sets U and V such that A⊆U and B⊆V.
(3) w-normal [12], if for any pair of disjoint w -closed sets A and B, there exists disjoint open sets U and V such that A⊆ U and
B⊆V.

Definition 2.4: [2] A topological space X is called T_{weakly} - space if every weakly-closed set in it is closed set.

Definition 2.5:A map f: (X, τ)⟶ (Y, τ) is said to be
(i)weakly-continuous map[11]if f^{-1}(V)is a weakly-closed set of (X, τ) for every closed set V of (Y, τ).
(ii)weakly-irresolute map[11]if f^{-1}(V)is a weakly-closed set of (X, τ) for everyweakly-closed set V of (Y, τ).

3Weakly Separation axioms in Regular Spaces

In this section, we introduce a new class of spaces called weakly-regular spaces using Weakly-closed sets and obtain some of their characterizations.

Definition 3.1. A topological space X is said to be weakly-regular if for each weakly closed set F and a point x∉F, there exist disjoint open sets G and H such that F⊆G andx ϵH.
We have the following interrelationship between weakly-regularity and regularity.

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Theorem 3.2. Every weakly-regular space is regular.

Proof: Let X be a weakly-regular space. Let F be any closed set in X and a point x∉Xsuch that x∉F. By [2], F is weakly-closed and x ∉F. Since X is a weakly-regular space, there exists a pair of disjoint open sets G and H such that F ⊆ G and x ϵH. Hence X is a regular space.

Remark 3.3. If X is a regular space and T_{weakly}space, then X is weakly regular We have the following characterization.

Theorem 3.4. The following statements are equivalent for a topological space X
(i) X is a weakly regular space
(ii) For each x ϵX and each weakly-open neighbourhood U of x there exists an open neighbourhood N of x such that cl(N)⊆U.

Proof: (i) implies(ii): Suppose X is a weakly regular space. Let U be any weakly neighbour-hood of x. Then there exists weakly open set G such that x ϵ G ⊆U. Now X –Gis weakly closed set and
x∉X - G. Since X is weakly regular, there exist open sets Mand N such that X-G⊆M,
x ϵ N and M∩N = $\varphi$and so N ⊆X-M. Now cl(N) ⊆ cl(X -M) = X -M and X -M ⊆M.
This implies X -M⊆ U. Therefore cl(N)⊆U.
(ii) implies(i): Let F be any weakly closed set in X and x ϵX -F and X - F is a Weakly-open and so X - F is a weakly-neighbourhood of x. By hypothesis, there existsan open neighbourhood N of x such that x ϵN and cl(N)⊆X - F. This implies F⊆X-cl(N) is an open set containing F and N ∩ f(X - cl(N)= φ . Hence X is weakly- regular space.
We have another characterization of weakly-regularity in the following.

Theorem 3.5: A topological space X is weakly-regular if and only if for each weakly-closed set F of X and each x ϵX - F there exist open sets G and H of X such that x ϵ G,F⊆H and cl(G) ∩ cl(H) = ∅.
Proof: Suppose X is weakly-regular space. Let F be a weakly-closed set in X with x ∉F.Then there exists open sets M and H of X such that x ϵ M, F ⊆H and M∩H =∅.This implies M∩cl(H) = ∅.As X is weakly-regular, there exist open sets U and V such that x ϵ U, cl(H)⊆V and U∩V = ∅. so cl(U)∩V = ∅.Let G = M ∩U, then G and H are open sets of X such that xϵG, F ⊆ H and cl(H) ∩cl(H) =∅.
Conversely, if for each weakly-closed set F of X and each x ϵ X -F there exists opensets G and H such that x ϵ G, F⊆ H and cl(H)∩cl(H) =∅.This implies x ϵG,F⊆H and G ∩ H = ∅. Hence X is weakly- regular.
Now we prove that weakly- regularity is a hereditary property.

Theorem 3.6. Every subspace of a weakly-regular space is weakly-regular.
Proof: Let X be a weakly- regular space. Let Y be a subspace of X. Let x ϵ Y and F bea weakly-closed set in Y such that x∉F. Then there is a closed set and so weakly-closed set A of X with F = Y ∩ A and x ∉A. Therefore we have x ϵ X, A is weakly– closed in X such that x∉A. Since X is weakly- regular, there exist open sets G and H such that x ϵG, A⊆H and G∩H = φ. Note that Y ∩G and Y∩H are open sets in Y .Also x ϵ G and x ϵY, which implies x ϵ Y ∩G and A ⊆ H implies Y∩ G⊆Y ∩H,F⊆Y ∩H. Also (Y ∩G)∩ (Y ∩ H) = φ. Hence Y is weakly-regular space.
We have yet another characterization of weakly-regularity in the following.

Theorem 3.7 : The following statements about a topological space X are equivalent:
(i) X is weakly-regular
(ii) For each x ϵ X and each weakly-open set U in X such that x ϵ U there exists an open set V in X such that x ϵ V⊆cl(V)⊆U.
(iii) For each point x ϵX and for each weakly-closed set A with x ∉ A, there exists an open set V containing x such that cl(V)∩A = φ.

Proof: (i)implies(ii): Follows from Theorem 3.5.
(ii) implies(iii): Suppose (ii) holds. Let x ϵ X and A be an weakly-closed set of X such that x ∉ A.Then X - A is a weakly-open set with x ϵX -A. By hypothesis, there exists an open set V such that x ϵV ⊆ cl(V) ⊆ X - A. That is xϵ V , V⊆ cl(A) andcl(A) ⊆ X - A. So x ϵV and cl(V)∩A = φ.
(iii) implies(i): Let x ϵ X and U be an weakly-open set in X such that x ϵ U. Then X - U is an weakly closed set and x∉ X - U. Then by hypothesis, there exists an open set V containing x such that cl(A) ∩(X -U) = Á. Therefore x ϵV , cl(V)⊆U sox ϵ V⊆ cl(V)⊆U.
The invariance of weakly-regularity is given in the following.

Theorem 3.8: Let f : X → Y be a bijective, weakly-irresolute and open map from a weakly- regular space X into a topological space Y , then Y is weakly-regular.
Proof: Let y ϵ Y and F be a weakly closed set in Y with y ∉ F. Since F is weakly- irresolute, f^{-1}(F) is weakly-closed set in X. Let f(x) = y so that x = f^{-1} (y) and x ∉ f^{-1}(F). Again X is weakly-regular space, there exist open sets U and V such that x ϵ U and f^{-1}(F) ⊆ G, U ∩ V = φ. Since f is open and bijective, we have y f(U),F ⊆ f(V) and f(U) ∩f(V) = f(U∩V) = f(φ) = φ . Hence Y is weakly-regular space.

Theorem 3.9. Let f : X → Y be a bijective, weakly-closed and open map from atopological space X into a weakly-regular space Y . If X is T_{weakly}space, then X is weakly-regular.

Proof: Let $x \epsilon X$ and F be an weakly-closed set in X with $x \notin F$. Since X is T_{weakly}space, F is closed in X. Then f(F) is weakly closed set with $f(x) \notin f(F)$ in Y , since f is weakly- closed. As Y is weakly-regular, there exist open sets U and V such that $x \epsilon$ Uandf(x) ϵ U and f(F) ⊆V . Therefore $x \epsilon f^{-1}(U)$ and $F \subseteq f^{-1}(V)$. Hence X is weakly-regular space.

Theorem 3.10. If f : X ⟶ Y is w-irresolute, continuous injection and Y is weakly-regular space, then X is weakly- regular.

Proof: Let F be any closed set in X with $x \notin F$. Since f is w-irresolute, f is weakly- closed set in Y and $f(x) \epsilon f(F)$. Since Y is weakly- regular, there exists open sets U and V such that $f(x) \epsilon U$ and
$f(F) \subseteq V$. Thus $x \epsilon f^{-1}(U), F \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \varphi$. Hence X is weakly- regular space.

4 Weakly Separation axioms in Normal Spaces

In this section, we introduce the concept of weakly normal spaces and study some of their characterizations.

Definition 4.1. A topological space X is said to be weakly-normal if for each pair of disjoint weakly- closed sets A and B in X, there exists a pair of disjoint open sets U and V in X such that A ⊆ U and B ⊆V
We have the following interrelationship.

Theorem 4.2. Every weakly-normal space is normal.
Proof: Let X be a weakly-normal space. Let A and B be a pair of disjoint closed sets in X. Since A and B are weakly- closed sets in X. Since X is weakly-normal, there exists a pair of disjoint open sets G and H in X such that A ⊆ G and B ⊆ H. Hence X is normal.

Remark 4.3. The converse need not be true in general as seen from the following example.

Example 4.4. Let X = Y ={a,b,c,d},τ ={X, Ø,{a},{c},{a,c},{b,c,d}}Then the space X is normal but not weakly- normal, since the pair of disjoint weakly- closed sets namely, A = {a,d}and B = {b,c}for which there do not exists disjoint open sets Gand H such that A ⊆ G and B⊆ H.

Remark 4.5.:If X is normal and T_{weakly}-space, then X is weakly-normal. Hereditary property of weakly-normality is given in the following.

Theorem 4.6. A weakly- closed subspace of a weakly-normal space is weakly-normal. We have the following characterization.

Theorem 4.7. The following statements for a topological space X are equivalent:
(i) X is weakly- normal
(ii) For each weakly- closed set A and each weakly- open set U such that
A⊆U, there exists an open set V such that A⊆V⊆cl(V)⊆U
(iii) For any weakly-closed sets A, B, there exists an open set V such that A⊆V and cl(V)∩B = φ.
(iv) For each pair A, B of disjoint weakly-closed sets there exist open sets U and V such that A ⊆ U,B ⊆V and cl(U) ∩cl(V) = φ.

Proof: (i) implies(ii): Let A be a weakly-closed set and U be a weakly-open set such that A⊆ U. Then A and X -U are disjoint weakly-closed sets in X. Since X is weakly-normal , there exists a pair of disjoint open sets V and W in X such that A ⊆ V and X -U⊆W. Now X -W ⊆ X -(X -U), so X -W⊆ U also V∩W = φ.implies V ⊆ X -W, socl(V) ⊆ cl(X -W) which implies cl(V) ⊆ X -W. Therefore cl(V) ⊆ X -W⊆ U. So cl(V) ⊆ U. Hence A ⊆ V ⊆ cl(V) ⊆ U.
(ii) implies(iii): Let A and B be a pair of disjoint weakly closed sets in X. Now A∩B = φ,so A⊆X -B, where A is weakly-closed an⊆d X - B is weakly-open . Then by (ii) there exists an open set V such that A⊆V⊆cl(V)⊆X - B. Now cl(V) ⊆X - B implies cl(V) ∩ B = φ . Thus A ⊆V and
cl(V) ∩ B = φ.
(iii) implies(iv): Let A and B be a pair of disjoint weakly-closed sets in X. Then from (iii)there exists an open set U such that A⊆U and cl(U) ∩B = φ. Since cl(V) is closed, so weakly-closed set. Therefore cl(V) and B are disjoint weakly closed sets in X. By hypothesis,
ther exists an open set V , such that B⊆V and cl(U) ∩cl(V) = φ.
(iv) implies(i): Let A and B be a pair of disjoint weakly-closed sets in X. Then from (iv)there exist an open sets U and V in X such that A⊆U, B⊆V and cl(U)∩cl(V) = φ.So A ⊆ U , B⊆V and U∩V = φ.Hence X weakly-normal.

Theorem 4.8. Let X be a topological space. Then X is weakly-normal if and only if for any pair A, B of disjoint weakly-closed sets there exist open sets U and V of X such that A⊆U,B⊆V and cl(U) ∩cl(V) = φ.

Theorem 4.9. Let X be a topological space. Then the following are equivalent:
(i) X is normal
(ii) For any disjoint closed sets A and B, there exist disjoint weakly- open sets U and V
such that A⊆U,B⊆V .
(iii) For any closed set A and any open set V such that A⊆ V, there exists an weakly-open set U of X such that A⊆U⊆αcl(U) ⊆ V .

Proof: (i) implies(ii): Suppose X is normal. Since every open set is weakly-open [2], (ii)follows.
(ii) implies(iii): Suppose (ii) holds. Let A be a closed set and V be an open set containing A. Then A and X - V are disjoint closed sets. By (ii), there exist disjoint weakly-

open sets U and W such that A⊆U and X - V ⊆ W, since X -V is closed, so weakly- closed. From [2], we have X - V⊆α-int(W) and U ∩ α-int(W) =φ.and so we have α-cl(U) ∩ α-int(W) = φ. Hence A ⊆ U ⊆α-cl(U) ⊆ X – α-int(W)⊆V. Thus A ⊆ U ⊆α-cl(U) ⊆ V .

(iii) implies(i): Let A and B be a pair of disjoint closed sets of X. Then A ⊆X - B and X -B is open. There exists a weakly- open set G of X such that A ⊆ G ⊆α-cl(G) ⊆X-B.Since A is closed, it is w- closed, we have A ⊆α-int(G). Take U = int(cl(int(α-int(G)))) and V = int(cl(int(X –α-cl(G)))). Then U and V are disjoint open sets of X such that A ⊆U and B ⊆ V. Hence X is normal.

We have the following characterization of weakly- normality and weakly- normality.

Theorem 4.10. Let X be a topological space. Then the following are equivalent:

(i) X is α-normal.

(ii) For any disjoint closed sets A and B, there exist disjoint weakly- open sets U and V such that A⊆U,B⊆V and U∩ V = φ.

Proof: (i) implies(ii): Suppose X is α- normal. Let A and B be a pair of disjoint closedsets of X. Since X is α-normal, there exist disjoint $\alpha -$ open sets U and V such that A⊆U and B⊆V and U ∩ V = φ.

(ii) implies(i):Let A and B be a pair of disjoint closed sets of X. The by hypothesis there exist disjoint weakly- open sets U and V such that A⊆U and B ⊆ V and U ∩V = φ .Sincefrom [2], A⊆α-intU and B ⊆$\alpha -$ int(V)and α –intU∩ α-intV = φ. Hence X is α-normal.

Theorem 4.11. Let X bea α- normal, then the following hold good:

(i)For each closed set A and every weakly- open set B such that A⊆B their exists a αopen set U such that A⊆U⊆α-cl(U)⊆ B.

(ii) For every weakly-closed set A and every open set B containing A, there exist a α-open set U such that A⊆U⊆ α-cl(U)⊆B.

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