Compactness and Connectedness in Weakly Topological Spaces

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Abstract: In this paper, we introduce and investigate topological spaces called W-compactness Spaces and W-connectedness space and we get several characterizations and some of their properties. Also we investigate its relationship with other types of functions.

Mathematics Subject Classification: 54D05, 54D30.

Keywords: *W-open set, W-closed sets, W-compact spaces, W-connectedness*

I. INTRODUCTION

The notions of compactness and connectedness are useful and fundamental notions of not only general topology but also of other advanced branches of mathematics. Many researchers have investigated the basic properties of compactness and connectedness. M. Ganster and M.Steiner [5] introduced and studied the properties of gb-closed sets in topological spaces. The aim of this paper is to introduce the concept of W-compactness and W-connectedness in topological spaces and is to give some characterizations of W-compact spaces.

II. PRELIMINARY NOTES

Throughout this paper (X, τ), (Y, σ) are topological spaces with no separation axioms assumed unless otheWise stated. Let A⊆X. The closure of A and the interior of A will be denoted by Cl(A) and Int(A) respectively.

Definition 2.1:A subset A of X is said to be b-open [1] if A⊆Int(Cl(A))∪Cl(Int(A)). The complement of b-open set is said to be b-closed. The familyof all b-open sets (respectively b-closed sets) of (X, τ) is denoted by bO(X, τ)[respectively bCL(X, τ)].

Definition 2.2: Let A be a subset of X. Then

(i) b-interior [1] of A is the union of all b-open sets contained in A.

(ii) b-closure [1] of A is the intersection of all b-closed sets containing A.The b-interior [respectively b-closure] of A is denoted by b-Int(A) [respectivelyb-Cl(A)].

Definition 2.3: Let A be a subset of X. Then A is said to be W-closed if cl(A) ⊆U whenever A ⊆U and U∈O(X, τ). The complement of W-closed set is called W-open. The family of all W-open [respectively W-closed] sets of (X, τ) is denoted by WO(X, τ) [respectively,W-CL(X, τ)].

Definition 2.4: The W-closure [35] of a set A, denoted by W-Cl(A), is the intersection of all W-closed sets containing A.

Definition 2.5: The W-interior [35] of a set A, denoted by W-Int(A), is the union of all W-open sets contained in A.

Remark 2.6: Every closed set is W-closed.

III. W-COMPACTNESS

Definition 3.1:A collection $\{A_i : i\in\wedge\}$ of W-open sets in a topological space X is called a W-open cover of a subset B of X if $B\subset\{A_i : i\in\wedge\}$ holds.

Definition 3.2:A topological space X is W-compact if every W-open cover of X has a finite sub-cover.

Definition 3.3:A subset B of a topological space X is said to be W-compact relative to X if, for every collection $\{A_i : i\in\wedge\}$ of W-open subsets of X such that $B \subset \cup \{A_i : i\in\wedge\}$ there exists a finite subset $\wedge_0$of ∧such that $B \subseteq \cup \{A_i : i\in\wedge_0\}$.

Definition 3.4:A subset B of a topological space X is said to be W-compact if B is W-compact as a subspace of X.

Theorem 3.5: Every W-closed subset of a W-compact space is W-compact relative to X.

Proof: Let A be W-closed subset of W-compact space X. Then $A^C$is W-open in X. Let $M = \{G_\alpha : \alpha\in\wedge\}$ be a cover of A by W-open sets in X. Then $M* = M\cup A^c$is a W-open cover of X. Since X is W-compact M∗is reducible to a finite subcover of X, say $X = G\alpha_1\cup G\alpha_2\cup\cdot\cdot\cdot\cup G\alpha_m\cup A^C$, $G\alpha_k\in M$. ButA and$A^C$are disjoint hence $A \subset G\alpha_1\cup\cdot\cdot\cdot\cup G\alpha_m$, $G\alpha_k\in M$, which impliesthat any W-open cover M of A contains a finite sub-cover.

Therefore A is W-compact relative to X.Thus every W-closed subset of a W-compact space X is W-compact.

Definition 3.6:A function f :X → Y is said to be W-continuous [5] if $f^{-1}(V)$ is W-closed in X for every closed set V of Y .

Definition 3.7:A function f :X → Y is said to be W-irresolute [5] if $f^{-1}(V)$ is W-closed in X for every W-closed set V of Y .

Theorem 3.8:A W-continuous image of a W-compact space is compact

Proof. Let f :X → Y be a W-continuous map from a W-compact space X onto a topological space Y. Let $\{A_i : i\in\wedge\}$ be an open cover of Y . Then $\{f^{-1}(A_i) : i\in\wedge\}$ is a W-open cover of X. Since X is W-compact it has a finite sub-cover say $\{f^{-1}(A_1), f^{-1}(A_2), \cdot\cdot\cdot, f^{-1}(A_n)\}$. Since f is onto $\{A_1, \cdot\cdot\cdot, An\}$ is a cover of Y , which is finite. Therefore Y is compact.

Theorem 3.9: If a map f :X → Y is W-irresolute and a subset B of X is W-compact relative to X, then the image f(B) is W-compact relative to Y .

Proof. Let $\{A_\alpha : \alpha \in\wedge\}$ be any collection of W-open subsets of Y such that $f(B) \subset\cup\{A_\alpha : \alpha \in\wedge\}$. Then $B \subset\cup\{f^{-1}(A_\alpha) : \alpha \in\wedge\}$ holds. Since by hypothesis B is W-compact relative to X there exists a finite subset $\wedge_0$of ∧ such that $B\subset\cup \{f^{-1}(A_\alpha) : \alpha \in\wedge_0\}$ Therefore we have $f(B) \subset\cup\{A_\alpha : \alpha \in\wedge_0\}$, which shows that f(B) is W compact relative to Y.

IV. W-CONNECTEDNESS

Definition 4.1: A topological space X is said to be W-connected if X cannot be expressed as a disjoint union of two non-empty W-open sets. A subsetof X is W-connected if it is W-connected as a subspace.

Example 4.2: Let X = {a, b} and let τ = {X, φ, {a}}. Then it is W-connected.

Remark 4.3: Every W-connected space is connected but the converse need not be true in general, which follows from the following example.

Example 4.4: Let X = {a, b} and let τ = {X, φ}. Clearly (X, τ) is connected. The W-open sets of X are {X, φ, {a} ,{b}}. Therefore (X, τ) is not a W-connected space, because X = {a} ∪ {b} where {a} and {b} are non-empty W-open sets.

Theorem 4.5: For a topological space X the following are equivalent.

(i) X is W-connected.

(ii) X and φ are the only subsets of X which are both W-open and W-closed.

(iii) Each W-continuous map of X into a discrete space Y with at least two points is a constant map.

Proof: (i) ⇒(ii) :

Let O be any W-open and W-closed subset of X. Then O^C is both W-open and W-closed. Since X is disjoint union of the W-open sets O and O^C implies from the hypothesis of (i) that either O = φ or O = X.

(ii) ⇒(i) :

Suppose that X = A∪B where A and B are disjoint non-empty W-open subsets of X. Then A is both W-open and W-closed. By assumption A = φ or X. Therefore X is W-connected.

(ii) ⇒(iii) :

Let f :X → Y be a W-continuous map. Then X is covered by W-open and W-closed covering

{f^{-1}(Y) : y ∈(Y)}. By assumption f^{-1}(y) = φ or X for each y ∈Y. If f−1(y) = φ for ally ∈Y , then f fails to be a map. Then there exists only one point y ∈Y such that f^{-1}(y) ≠φ and hence

f^{-1}(y) = X. This shows that f is a constant map.

(iii) ⇒(ii) :

Let O be both W-open and W-closed in X. Suppose O ≠φ. Let f :X → Y be a W-continuous map defined by f(O) = y and f(O^C) = {w} for somedistinct points y and w inY .

By assumption f is constant. Therefore we have O = X.

Theorem 4.6: If f :X → Y is a W-continuous and X is W-connected, then Y is connected.

Proof: Suppose that Y is not connected. Let Y = A∪B where A and B are disjoint non-empty open set in Y . Since f is W-continuous and onto, X = f^{-1}(A) ∪f^{-1}(B) where f^{-1}(A) and f^{-1}(B) are disjoint non-empty W-opensets in X. This contradicts the fact that X is W-connected. Hence Y is connected.

Theorem 4.7: If f :X → Y is a W-irresolute surjection and X is W-connected, then Y is W-connected.

Proof: Suppose that Y is not W-connected. Let Y =A∪B where A andB are disjoint non-empty W-open set in Y. Since f is W-irresolute and onto, X = f^{-1}(A)∪f^{-1}(B) where f^{-1}(A) andf^{-1}(B) are disjoint non-empty W-open sets in X. This contradicts the fact that X is W-connected. Hence Y is connected.

Theorem 4.8: In a topological space (X, τ) with at least two points, if W-O(X, τ) = W-CL(X, τ) then X is not W-connected.

Proof: By hypothesis we have W-O(X, τ) = W-CL(X, τ) and by Remark 2.6 we have every closed set is W-closed, there exists some non-empty proper subset of X which is both W-open and W-closed in X. So by last Theorem 4.5 we have X is not W-connected.

Definition 4.9: A topological space X is said to be T_W-space if every W-closed subset of X is closed subset of X.

Theorem 4.10: Suppose that X is a T_W-space then X is connected if and only if it is W-connected.

Proof: Suppose that X is connected. Then X can not be expressed as disjoint union of two non-empty proper subsets of X. Suppose X is not a W-connected space. Let A and B be any two W-open subsets of X such that X = A∪B, where A∩B = φ and A⊂X, B⊂X. Since X is T_W-space and A,Bare W-open,A,Bare open subsets of X, which contradicts that X is connected.Therefore X is W-connected. Conversely, every open set is W-open. Therefore every W-connected space is connected.

Theorem 4.11: If the W-open sets C and D form a separation of X and if Y is W-connected subspace of X, then Y lies entirely within C or D.

Proof: Since C and D are both W-open in X the sets C∩Y and D∩Y are W-open in Y these two sets are disjoint and their union is Y. If they wereboth non-empty, they would constitute a separation of Y. Therefore, one ofthem is empty. Hence Y must lie entirely in C or in D.

Theorem 4.12: Let A be a W-connected subspace of X. If A⊂B⊂W-Cl(A) then B is also W-connected.

Proof: Let A be W-connected and let A⊂B⊂W-Cl(A). Suppose that B = C∪D is a separation of B by W-open sets. Then by Theorem 4.11 above A must lie entirely in C or in D.Suppose that A ⊂C, then W-Cl(A) ⊆W-Cl(C). Since W-Cl(C) and D are disjoint, B cannot intersect D. This contradicts the fact that D is non-empty subset of B. So D = φ which implies B is W-connected.

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Compactness and W-connectedness in topological spaces 475.

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