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MINIMAL WEAKLY HOMEOMORPHISM AND MAXIMAL WEAKLY HOMEOMORPHISM IN TOPOLOGICAL SPACES

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Abstract

In this paper a new classes of homeomorphism called minimal weakly homeomorphism and maximal weakly homeomorphism are introduced and investigated. Further, some properties related to these new concepts are studied by extending the definitions related to continuous maps and homeomorphism in topological spaces. [1]

1 INTRODUCTION

In the year 2001 and 2003, F.Nakaoka and N.oda [1, 2, 3] introduced and studied minimal open (resp. minimal closed) sets which are subclass of open (resp.closed) sets. The family of all minimal open (minimal closed) in a topological space X is denoted by mio(X) (mic(X)). Similarly the family of all maximal open (maximal closed) sets in a topological space X is denoted by MaO(X) (MaC(X)). The complements of minimal open sets and maximal open sets are called maximal closed sets and minimal closed sets respectively. In the year 2000, M.Sheik John [4] introduced and studied weakly homeomorphism in topological spaces and in the year 2008 B.M.Ittanagi [5] introduced and studied minimal open sets and maps in topological spaces and minimal homeomorphism and maximal homeomorphism in topological spaces. In the year 2014, R.S.Wali and Vivekananda Dembre [6,7] introduced and studied minimal weakly open sets and maximal weakly closed sets and maximal weakly open sets and minimal weakly closed sets in topological spaces. In the year 2014 [8] Vivekananda Dembre, Manjunath Gowda and Jeetendra Gurjar introduced and studied minimal weakly and maximal weakly continuous functions in topological spaces. In the year 2014 [9] Vivekananda Dembre and Jeetendra Gurjar introduced and studied minimal weakly and maximal weakly open maps in topological spaces. In this paper some concepts related to continuous maps and homeomorphism in topological spaces are introduced and some properties related to these concepts are studied in the form of theorems.

[1]Keywords: fuzzy set, fuzzy soft set, fuzzy soft multi-set, fuzzy soft multi point, fuzzy soft multi topological space,sequence.

2 SOME KNOWN DEFINITIONS

In this section some known definitions related to open and closed sets , continuous function and homeomorphism in a topological space are given. These definitions are used in proving the theorems discussed in Section 3.

Definition 2.1 *A proper non-empty open subset U of a topological space X is said to be minimal open set if any open set which is contained in U is φ or U.*

Definition 2.2 *A proper non-empty open subset U of a topological space X is said to be maximal open set if any open set which is contained in U is X or U.*

Definition 2.3 *A proper non-empty closed subset F of a topological space X is said to be minimal closed set if any closed set which is contained in F is φ or F.*

Definition 2.4 *A proper non-empty closed subset F of a topological space X is said to be maximal closed set if any closed set which is contained in F is X or F.*

Definition 2.5 *Let X and Y be the topological spaces. A bijective function $f : X \to Y$ is called weakly homeomorphism if both f and f^{-1} are weakly continuous.*

Definition 2.6 *Let X and Y be the topological spaces. A bijective function $f : X \to Y$ is called*

(i) *Minimal homeomorphism if both f and f^{-1} are minimal continuous maps.*

(ii) *Maximal homeomorphism if both f and f^{-1} are maximal continuous maps.*

Definition 2.7 *Let X and Y be the topological spaces. A map $f : X \to Y$ is called*

(i) *Minimal continuous function if for every minimal open set N in Y, $f^{-1}(N)$ is an open set in X.*

(ii) *Maximal continuous function if for every maximal open set N in Y, $f^{-1}(N)$ is an open set in X.*

(iii) *Minimal open map if for every open set U of X, f(U) is minimal open set in Y.*

(iv) *Maximal open map if for every open set U of X, f(U) is maximal open set in Y.*

(v) *Minimal closed map if for every closed set F of X, f(F) is minimal closed set in Y.*

(vi) *Maximal closed map if for every closed set F of X, f(F) is maximal closed set in Y.*

(vii) *Minimal-Maximal open map if for every minimal open set N of X, f(N) is maximal open set in Y.*

(viii) *Maximal-Minimal open map if for every maximal open set N of X,f(N) is minimal open set in Y.*

(ix) *Minimal-Maximal continuous if for every minimal open set N in Y, $f^{-1}(N)$ is a maximal open set in X.*

(x) *Maximal-Minimal continuous if for every maximal open set N in Y, $f^{-1}(N)$ is a minimal open set in X.*

Definition 2.8 *Let X and Y be two topological spaces. A map $f : X \to Y$ is called*

(i) *Minimal weakly continuous if for every minimal weakly open set N in Y,$f^{-1}(N)$ is an open set in X.*

(ii) *Maximal weakly continuous if for every maximal weakly open set N in Y,$f^{-1}(N)$ is an open set in X.*

Definition 2.9 *A proper non-empty weakly open subset U of X is said to be minimal weakly open set if any weakly open set which is contained in U is slashed O or U.*

Definition 2.10 *A proper non-empty weakly closed subset U of X is said to be maximal weakly open set if any weakly open set which is contained in U is X or U.*

Definition 2.11 *Let X and Y be the topological spaces. A map $f : X \to Y$ is called*

(i) *minimal weakly open map if for every open set U of X , f(U) is minimal weakly open set in Y.*

(ii) *maximal weakly open map if for every open set U of X , f(U) is maximal weakly open set in Y.*

(iii) *minimal weakly closed map if for every closed set F of X , f(F) is minimal weakly closed set in Y.*

(iv) *maximal weakly closed map if for every closed set F of X , f(F) is maximal weakly closed set in Y..*

3 MINIMAL WEAKLY AND MAXIMAL WEAKLY HOMEOMORPHISM

We prove certain theorems related to continuous maps and homeomorphism in a topological space by introducing new concepts. For each theorem we also provide a counter example, if the converse part is not true. We begin with

Definition 3.1 *A bijection function $f : X \to Y$ is called*

(i) *Minimal weakly homeomorphism if both f and f^{-1} are minimal weakly continuous maps.*

(ii) *Maximal weakly homeomorphism if both f and f^{-1} are maximal weakly continuous maps.*

Theorem 3.1 *Every homeomorphism is minimal weakly homeomorphism but not conversely.*

Proof: Let f : X→Y be a homeomorphism. Now f and $f^{-1}$are continuous maps then f and $f^{-1}$are minimal weakly continuous as every continuous map is minimal weakly continuous. Hence f is a minimal weakly homeomorphism.

Example 3.1 *Let X=Y=$\{a,b,c\}$, $\tau = \{ X, \varphi, \{a\},\{a,b\}\}$ and , $\mu = \{ Y, \varphi, \{a\},\{b\},\{a,b\} \}$. Define f : X→Y by f(a)=a, f(b)=a and f(c)=c. Then f and f^{-1} are minimal weakly continuous maps. Hence f is a minimal weakly homeomorphism but not a homeomorphism. Since f is not continuous map for the open set $\{b\}$ in Y ; f^{-1} ($\{b\}$) = $\{b\}$ which is not open set in X.*

Theorem 3.2 *Every homeomorphism is maximal weakly homeomorphism but not conversely.*

Proof: Let f : X→Y be a homeomorphism. Now f and $f^{-1}$are continuous maps then f and $f^{-1}$are maximal weakly continuous as every continuous map is maximal weakly continuous. Hence f is a maximal weakly homeomorphism.

Example 3.2 *Let X=Y=$\{a,b,c\}$, $\tau = \{X, \varphi,\{a,b\}\}$ and $\mu = \{ Y, \varphi, \{a\},\{b\},\{a,b\} \}$. Let f : X→Y be a identity function, then f and f^{-1} are maximal weakly continuous maps. Hence f is a maximal weakly homeomorphism but not a homeomorphism, Since f is not continuous map for the open set $\{b\}$ in Y ; f^{-1} ($\{b\}$) = $\{b\}$ which is not open set in X.*

Remark 3.1 *Minimal weakly homeomorphism and maximal weakly homeomorphism are independent of each other.*

Example 3.3 *In example (3.1) f is minimal weakly homeomorphism but not maximal weakly homeomorphism . In example (3.2) f is maximal weakly homeomorphism but not minimal weakly homeomorphism.*

Theorem 3.3 *Let f : X→Y be a bijective and minimal weakly continuous then the following statements are equivalent.*

(i) *f : X→Y is minimal weakly homeomorphism.*

(ii) *f is minimal weakly open map.*

(iii) *f is maximal weakly closed map.*

Proof: **(i)⟹(ii):**Let N be any minimal weakly open set in X ; by assumption $(f^{-1})^{-1}$(N) is an open set in Y. But $(f^{-1})^{-1}$(N) = f(N) is an open set in Y ; therefore f is a minimal weakly open map.
(ii)⟹(iii): Let F be any maximal weakly closed set in X ; then X - F is a minimal weakly open set in X ; by assumption f (X - F)is an open set in Y. But f (X - F)= Y - f (F) is an open set in Y ; therefore f(F) is a closed set in Y. Hence f is a maximal weakly closed map.
(iii)⟹(i): Let N be any minimal weakly open set in X ; then X - N is a maximal weakly closed set in X ; by assumption f(X-N)is an closed set in Y. But f(X - N) = (f^{-1})(X - N) = Y - (f^{-1})(N) is closed set in Y ; therefore (f^{-1})(N) is an open set in Y. Hence f^{-1}:Y→X is a minimal weakly homeomorphism and similarly we can show that f is minimal weakly homeomorphism.

Theorem 3.4 *Let f : X→Y be a bijective and maximal weakly continuous then the following statements are equivalent.*

(i) *f : X→Y is maximal homeomorphism.*

(ii) *f is maximal weakly open map.*

(iii) *f is minimal weakly closed map*

Proof: **(i)⟹(ii):** Let N be any maximal weakly open set in X ; by assumption $(f^{-1})^{-1}(N)$ is an open set in Y. But $(f^{-1})^{-1}(N)$ = f(N) is an open set in Y ; therefore f is a maximal weakly open map.
(ii)⟹(iii): Let F be any minimal weakly closed set in X ; then X-F is a maximal weakly open set in X ; by assumption f (X-F)is an open set in Y. But f (X-F)= Y - f (F) is an open set in Y ; therefore f(F) is a closed set in Y. Hence f is a minimal weakly closed map.
(iii)⟹(i): Let N be any maximal weakly open set in X ; then X - N is a minimal weakly closed set in X ; by assumption f(X-N) is closed set in Y. But f(X-N) = (f^{-1})(X -N) = Y - (f^{-1})(N) is closed set in Y ; therefore (f^{-1})(N) is an open set in Y. Hence f^{-1}:Y→X is a maximal weakly homeomorphism and similarly f is maximal weakly homeomorphism.

Definition 3.2 *:Let X and Y be two topological spaces. A map f : X→Y is called*

(i) *Minimal-Maximal weakly continuous if for every minimal weakly open set N in Y, $f^{-1}(N)$ is a maximal weakly open set in X.*

(ii) *Maximal-Minimal weakly continuous if for every maximal weakly open set N in Y, $f^{-1}(N)$ is a minimal weakly open set in X.*

(iii) *Minimal-Maximal weakly open map if for every minimal weakly open set N of X, f(N) is maximal weakly open set in Y.*

(iv) *Maximal-Minimal weakly open map if for every maximal weakly open set N of X, f(N) is minimal weakly open set in Y.*

(v) *Minimal-Maximal weakly closed map if for every minimal weakly closed set F of X, f(F) is maximal weakly closed set in Y.*

(vi) *Maximal-Minimal weakly closed map if for every maximal weakly closed set F of X, f(F) is minimal weakly closed set in Y.*

Definition 3.3 *A bijection f : X→Y is called*

(i) *min - max weakly homeomorphism if both f and $f^{-1}(N)$ are min - max weakly continuous maps.*

(ii) *max- min weakly homeomorphism if both f and $f^{-1}(N)$ are max - min weakly continuous maps.*

Theorem 3.5 *Every min-max weakly homeomorphism is minimal weakly homeomorphism but not conversely.*

Proof: Let f : X→Y be a min-max weakly homeomorphism. Now f and $f^{-1}$are continuous maps then f and $f^{-1}$are minimal weakly continuous as every continuous map is minimal weakly continuous. Hence f is a minimal weakly homeomorphism.

Example 3.4 *Let* (X, τ) *and* (Y, μ) *be two topological spaces where* $X = Y = \{a, b, c\}$, $\tau = \{X, \varphi, \{a\}, \{a, b\}\}$ *and* $\mu = \{Y, \varphi, \{a\}, \{b\}, \{a, b\}\}$. *Let f : X→Y be a function defined by f(a)=a, f(b)=a and f(c)=c, then f and* f^{-1} *are minimal weakly continuous maps.Hence f is a minimal weakly homeomorphism but not a min-max weakly homeomorphism. Since f is not a min-max weakly continuous map for the minimal weakly open set* $\{b\}$ *in Y,* $f^{-1}(b) = b$ *which is not a minimal weakly open set in X.*

Theorem 3.6 *Every max-min weakly homeomorphism is maximal weakly homeomorphism but not conversely.*

Proof: Let f : X→Y be a homeomorphism. Now f and $f^{-1}$are continuous maps then f and $f^{-1}$are maximal weakly continuous as every continuous map is maximal weakly continuous. Hence f is a maximal weakly homeomorphism.

Theorem 3.7 *Let f : X→Y be a bijective and min-max weakly continuous map then the following statements are equivalent.*

(i) *f is minimal - maximal weakly homeomorphism.*

(ii) *f is minimal - maximal weakly open map.*

(iii) *f is maximal - minimal weakly closed map.*

Proof: (i)⟹(ii): Let N be any minimal weakly open set in X ; by assumption $(f^{-1})^{-1}(N)$ is an open set in Y. But $(f^{-1})^{-1}(N) = f(N)$ is a maximal weakly open set in Y ; therefore f is a min-max weakly open map.
(ii)⟹(iii): Let F be any maximal weakly closed set in X ; then X-F is a minimal weakly open set in X ; by assumption f (X-F)is a maximal weakly open set in Y. But f (X-F)= Y- f (F) is a maximal weakly open set in Y ; therefore f(F) is a minimal weakly closed set in Y. Hence f is a max-min weakly closed map.
(iii)⟹(i): Let N be any minimal weakly open set in X ; then X - N is a maximal weakly closed set in X ; by assumption f(X-N)is a minimal weakly closed set in Y. But f(X-N) = (f^{-1})(X-N) = Y - (f^{-1})(N) is a minimal weakly open set in Y ; therefore (f^{-1})(N)is a maximal weakly open set in Y. Hence f^{-1}:Y→ X is a minimal-maximal weakly homeomorphism and similarly f is min-max weakly homeomorphism.

Theorem 3.8 *Let f : X→Y be a bijective and min-max weakly continuous map then the following statements are equivalent.*

(i) *f is max - min weakly homeomorphism.*

(ii) *f is max - min weakly open map.*

(iii) *f is min - max weakly closed map.*

Proof: (i)$\Longrightarrow$(ii): Let N be any maximal weakly open set in X ; by assumption $(f^{-1})^{-1}(N)$ is an open set in Y. But $(f^{-1})^{-1}(N) = f(N)$ is a minimal weakly open set in Y ; therefore f is a max-min weakly open map.
(ii)$\Longrightarrow$(iii): Let F be any minimal weakly closed set in X ; then X-F is a maximal weakly open set in X ; by assumption f (X-F)is a minimal weakly open set in Y. But f (X-F)= Y- f (F) is a minimal weakly open set in Y ; therefore f(F) is a minimal weakly closed set in Y. Hence f is a min-max weakly closed map.
(iii)$\Longrightarrow$(i): Let N be any maximal weakly open set in X ; then X - N is a minimal weakly closed set in X ; by assumption f(X-N)is a maximal weakly closed set in Y. But $f(X\text{-}N) = (f^{-1})(X\text{-}N) = Y - (f^{-1})(N)$ is a maximal weakly open set in Y ; therefore $(f^{-1})(N)$is a maximal weakly open set in Y. Hence f^{-1}:Y$\rightarrow$ X is a maximal-minimal weakly homeomorphism and similarly f is max-min weakly homeomorphism.

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