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# On the Wiener index of middle graph and its complement

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ABSTRACT

The *Wiener index* of a graph $G$ denoted by $W(G)$ is the sum of distances between all (unordered) pairs of vertices of $G$. In practice $G$ corresponds to what is known as the *molecular graph* of an organic compound. In this paper we obtain the Wiener index of middle graph and its complement for some standard class of graphs, we give bounds for Wiener index of middle graph and its complement also establish Nordhaus-Gaddum type of inequality for it.

## 1. Introduction

Let $G$ be a simple, connected, undirected graph with vertex set $V(G)=\{v_1,v_2,...,v_n\}$ and edge set $E(G)=\{e_1,e_2,...,e_m\}$. The distance between two vertices $v_i$ and $v_j$, denoted by $d(v_i,v_j)$ is the length of the shortest path between the vertices $v_i$ and $v_j$ in $G$. The shortest $v_i-v_j$ path is often called *geodesic*. The *diameter* $diam(G)$ of a connected graph $G$ is the length of any longest geodesic. The *degree* of a vertex $v_i$ in $G$ is the number of edges incident to $v_i$ and is denoted by $d_i = deg(v_i)$ [16].

The *Wiener index* (or *Wiener number*)[26] of a graph $G$, denoted by $W(G)$ is the sum of distances between all (unordered) pairs of vertices of $G$.

$$W(G)=\sum_{i<j} d(v_i,v_j)$$

The *Wiener index* $W(G)$ of the graph $G$ is also defined by

$$W(G)=\frac{1}{2}\sum_{v_i,v_j \in V(G)} d(v_i,v_j)$$

where the summation is over all possible pairs $v_i,v_j \in V(G)$.

The Wiener index is of certain importance in chemistry [13, 14]. It is one of the oldest and most thoroughly studied graph-based molecular structure-descriptors, so called *topological indices* [13, 23, 26]. Numerous of its chemical applications were reported [2, 11, 12, 14, 19, 22] and its mathematical properties are reasonably well understood [3, 5, 7, 8, 10, 17, 18, 25, 32].

Line graphs, total graphs and middle graphs are widely studied transformation graphs. Let $G=(V(G),E(G))$ be a graph. The *line graph* $L(G)$ of $G$ is the graph whose vertex set is $E(G)$ in which two vertices are adjacent if and only if they are adjacent in $G$. The *middle graph* $M(G)$ [29] of $G$ is the graph whose vertex set is $V(G)\cup E(G)$ in which two vertices $x$ and $y$ are adjacent if and only if at least one of $x$ and $y$ is an edge of $G$, and they are adjacent or incident in $G$. The number of vertices and edges in $M(G)$ are $n+m$ and $m+\frac{1}{2}\sum_{i=1}^{n} d_i^2$ [15],respectively. The

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*complement* of $G$, denoted by $\overline{G}$, is the graph with the same vertex set as $G$, where two vertices are adjacent if and only if they are not adjacent in $G$. We denote the *complement of middle graph* $M(G)$ [29] of $G$ by $\overline{M}(G)$. Its vertex set is $V(G) \cup E(G)$ in which the vertices $x$ and $y$ are joined by an edge if one of the following conditions holds:(i) $x, y \in V(G)$, (ii) $x, y \in E(G)$, and $x$ and $y$ are not adjacent in $G$, (iii) One of $x$ and $y$ is in $V(G)$ and the other is in $E(G)$, and they are not incident in $G$. Many papers are devoted to middle graph [1, 15, 20, 29].

In the following we denote by $C_n$, $P_n$, $S_n$ and $K_n$ the cycle , the path, the star, and the complete graph of order $n$, respectively. Other undefined notation and terminology can be found in [16].

The following theorems are useful for proving our main results.

**Theorem 1.1** [24]. *Let $G$ be connected graph with $n$ vertices and $m$ edges.*

If $diam(G) \le 2$, then $W(G) = n(n-1) - m$.

**Theorem 1.2** [4]. *For every tree $T$ of order $n$,*

$$W(L(T)) = W(T) - \binom{n}{2}.$$

**Theorem 1.3** [27]. *If $P_n$ is a path of order $n$, then*

$$W(P_n) = \frac{n(n^2-1)}{6}.$$

**Theorem 1.4** [27]. *If $S_n$ is a star of order $n$, then* $W(S_n) = (n-1)^2$.

**Theorem 1.5** [9]. *For every tree $T$ of order $n$,* $W(S_n) \le W(T) \le W(P_n)$.

**Theorem 1.6** [6]. *Let $G$ be a connected graph with $n$ vertices and $m$ edges.*

Then $\sum_{i=1}^{n} d_i^2 = m[\frac{2m}{n-1} + n - 2]$ if and only if $G$ is a star graph or a complete graph.

**Theorem 1.7** [28]. *Let $G$ be a connected graph with minimum degree $\delta(G) \ge 2$. Then $W(G) \le W(L(G))$. Equality holds only for cycles.*

**Theorem 1.8** [27]. *If $C_n$ is a cycle of order $n$, then*

$$W(C_n) = \begin{cases} \frac{n^3}{8} & \text{if } n \text{ is even} \\ \frac{n^3 - n}{8} & \text{if } n \text{ is odd.} \end{cases}$$

## 2. Results

**Theorem 2.1** *Let $T$ be a tree of order $n$ and Wiener index $W(T)$. Then*

$$W(M(T)) = 4W(T).$$

*Proof.* Let the vertex set of $T$ be $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_1', v_2', \ldots, v_{n-1}'\}$. $M(T)$ is a middle graph of $T$ with vertex set $\{v_1, v_2, \ldots, v_n\} \cup \{v_1', v_2', \ldots, v_{n-1}'\}$, where $v_1', v_2', \ldots, v_{n-1}'$ are vertices in $M(T)$ corresponding to edges of $T$.

Splitting the summation of the Wiener index of $M(T)$ into four parts,

$W(M(T))$ =half of the shortest distance between the vertices of $v_i$ and $v_j'$

+ half of the shortest distance between the vertices of $v_j'$ and $v_i$

+ half of the shortest distance between the vertices of $v_i$ and $v_j$

+ half of the shortest distance between the vertices of $v_i'$ and $v_j'$.

$$= \frac{1}{2}\left\{\begin{array}{l} d(v_1, v_1') + d(v_1, v_2') + \ldots + d(v_1, v_{n-1}') \\ + d(v_2, v_1') + d(v_2, v_2') + \ldots + d(v_2, v_{n-1}') \\ + \ldots + \ldots + \ldots\ldots\ldots\ldots\ldots + \\ + d(v_n, v_1') + d(v_n, v_2') + \ldots + d(v_n, v_{n-1}') \end{array}\right\}$$

$$+\frac{1}{2}\begin{Bmatrix} d(v_1',v_1)+d(v_1',v_2)+...+d(v_1',v_n) \\ +d(v_2',v_1)+d(v_2',v_2)+...+d(v_2',v_n) \\ +...+...+...............+ \\ +d(v_{n-1}',v_1)+d(v_{n-1}',v_2)+...+d(v_{n-1}',v_n) \end{Bmatrix}$$

$$+\frac{1}{2}\begin{Bmatrix} d(v_1,v_1)+d(v_1,v_2)+...+d(v_1,v_n) \\ +d(v_2,v_1)+d(v_2,v_2)+...+d(v_2,v_n) \\ +...+...+...............+ \\ +d(v_n,v_1)+d(v_n,v_2)+...+d(v_n,v_n) \end{Bmatrix}$$

$$+\frac{1}{2}\begin{Bmatrix} d(v_1',v_1')+d(v_1',v_2')+...+d(v_1',v_{n-1}') \\ +d(v_2',v_1')+d(v_2',v_2')+...+d(v_2',v_{n-1}') \\ +...+...+...............+ \\ +d(v_{n-1}',v_1')+d(v_{n-1}',v_2')+...+d(v_{n-1}',v_{n-1}') \end{Bmatrix}$$

$$=\frac{1}{2}\{2W(T)+2W(T)+2[W(T)+\binom{n}{2}]+2[W(T)-\binom{n}{2}]\}$$

$$W(M(T))=4W(T)$$

**Corollary 2.2** *If $P_n$ is a path of order $n$, then*

$$W(M(P_n))=\frac{2n(n^2-1)}{3}.$$

*Proof.* From Theorem 2.1, $W(M(T))=4W(T)$

$W(M(P_n))=4W(P_n)$.

From Theorem 1.3, $W(P_n)=\frac{n(n^2-1)}{6}$ and hence

$$W(M(P_n))=\frac{2n(n^2-1)}{3}\cdot$$

**Corollary 2.3** *If $S_n$ is a star graph of order $n$, then*

$$W(M(S_n))=4(n-1)^2.$$

*Proof.* From Theorem 2.1, $W(M(T))=4W(T)$

$W(M(S_n))=4W(S_n)$.

From Theorem 1.4, $W(S_n)=(n-1)^2$ and hence

$W(M(S_n))=4(n-1)^2\cdot$

**Corollary 2.4** *For any tree $T$ of order $n$,* $W(M(S_n))\le W(M(T))\le W(M(P_n))$.

*Proof.* From Theorem 1.5,

$$W(S_n)\le W(T)\le W(P_n)$$

Therefore $4W(S_n)\le 4W(T)\le 4W(P_n)$

From Theorem 2.1, Corollaries 2.2 and 2.3.

$$W(M(S_n))\le W(M(T))\le W(M(P_n)).$$

**Theorem 2.5** *If $K_n$ is a complete graph of order $n$, then*

$$W(M(K_n))=\frac{n(n^3+n-2)}{4}.$$

*Proof.* Let $K_n$ be a complete graph with $n$ vertices and $m$ edges. Then $M(K_n)$ has

$n_1=n+m$ vertices and $m_1=m+\frac{1}{2}\sum_{i=1}^{n}d_i^2$ edges.

In $M(K_n)$, distance between adjacent vertices is one and distance between nonadjacent vertices is two, therefore $diam(M(K_n))=2$.

From Theorem 1.1,

$$W(M(K_n))=n_1(n_1-1)-m_1 \quad (1)$$

$$\text{But } n_1=n+m=n+\binom{n}{2}=\frac{n^2+n}{2} \quad (2)$$

and $m_1=m+\frac{1}{2}\sum_{i=1}^{n}d_i^2$.

From Theorem 1.6, we have $\sum_{i=1}^{n} d_i^2 = n(n-1)^2$

$$m_1 = \binom{n}{2} + \frac{n(n-1)^2}{2} = \frac{n^3-n^2}{2} \qquad (3)$$

From equations (1),(2) and (3)

$$W(M(K_n)) = \frac{n^2+n}{2}\left(\frac{n^2+n}{2}-1\right)-\left(\frac{n^3-n^2}{2}\right)$$

$$W(M(K_n)) = \frac{n(n^3+n-2)}{4}.$$

**Theorem 2.6** *If $C_n$ is a cycle of order $n$, then*

$$W(M(C_n)) = \frac{n^3+2n^2-n}{2}.$$

*Proof.* Let vertex set of $C_n$ be $\{v_1, v_2, ..., v_n\}$ and edge set $\{v_1', v_2', ..., v_n'\}$. $M(C_n)$ is a middle graph of $C_n$ with vertex set $\{v_1, v_2, ..., v_n\} \cup \{v_1', v_2', ..., v_n'\}$, where $v_1', v_2', ..., v_n'$ are vertices in $M(C_n)$ corresponding to edges of $C_n$.

Splitting the summation of the Wiener index of $M(C_n)$ into four parts,

$W(M(C_n))=$

half of the shortest distance between the vertices of $v_i$ and $v_j'$

+ half of the shortest distance between the vertices of $v_j'$ and $v_i$

+ half of the shortest distance between the vertices of $v_i$ and $v_j$

+ half of the shortest distance between the vertices of $v_i'$ and $v_j'$.

$$= \frac{1}{2}\left\{\begin{array}{l} d(v_1,v_1')+d(v_1,v_2')+...+d(v_1,v_n') \\ +d(v_2,v_1')+d(v_2,v_2')+...+d(v_2,v_n') \\ +...+...+..............+ \\ +d(v_n,v_1')+d(v_n,v_2')+...+d(v_n,v_n') \end{array}\right\}$$

$$+ \frac{1}{2}\left\{\begin{array}{l} d(v_1',v_1)+d(v_1',v_2)+...+d(v_1',v_n) \\ +d(v_2',v_1)+d(v_2',v_2)+...+d(v_2',v_n) \\ +...+...+..............+ \\ +d(v_n',v_1)+d(v_n',v_2)+...+d(v_n',v_n) \end{array}\right\}$$

$$+ \frac{1}{2}\left\{\begin{array}{l} d(v_1,v_1)+d(v_1,v_2)+...+d(v_1,v_n) \\ +d(v_2,v_1)+d(v_2,v_2)+...+d(v_2,v_n) \\ +...+...+..............+ \\ +d(v_n,v_1)+d(v_n,v_2)+...+d(v_n,v_n) \end{array}\right\}$$

$$+ \frac{1}{2}\left\{\begin{array}{l} d(v_1',v_1')+d(v_1',v_2')+...+d(v_1',v_n') \\ +d(v_2',v_1')+d(v_2',v_2')+...+d(v_2',v_n') \\ +...+...+..............+ \\ +d(v_n',v_1')+d(v_n',v_2')+...+d(v_n',v_n') \end{array}\right\}$$

Case (1). $C_n$ is an odd cycle.

$$W(M(C_n)) = \frac{1}{2}[2W(C_n)+n\,diamM(C_n)]+\frac{1}{2}[2W(C_n)$$

$$+n\,diamM(C_n)]+\frac{1}{2}[2W(C_n)+n(n-1)]+\frac{1}{2}[2W(L(C_n))].$$

From Theorem 1.7, $W(L(C_n)) = W(C_n)$

$$W(M(C_n)) = \frac{1}{2}\{8W(C_n)+2n\,diamM(C_n)+n(n-1)\}$$

$$= 4W(C_n)+n^2.$$

From Theorem 1.8, we have $W(C_n) = \frac{n^3-n}{8}$

$$W(M(C_n)) = 4\frac{(n^3-n)}{8}+n^2$$

$$W(M(C_n)) = \frac{n^3+2n^2-n}{2}.$$

Case (2). $C_n$ is an even cycle.

$$W(M(C_n)) = \frac{1}{2}[2W(C_n) + n\,diam(C_n)] + \frac{1}{2}[2W(C_n)$$

$$+ n\,diam(C_n)] + \frac{1}{2}[2W(C_n) + n(n-1)] + \frac{1}{2}[2W(L(C_n))].$$

From Theorem 1.7, $W(L(C_n)) = W(C_n)$.
Therefore

$$W(M(C_n)) = \frac{1}{2}\{8W(C_n) + 2n\,diam(C_n) + n(n-1)\}$$

$$W(M(C_n)) = 4W(C_n) + n^2 - \frac{n}{2}.$$

From Theorem 1.8, we have $W(C_n) = \frac{n^3}{8}$

$$W(M(C_n)) = \frac{n^3 + 2n^2 - n}{2}.$$

From above two Cases, we have

$$W(M(C_n)) = \frac{n^3 + 2n^2 - n}{2}.$$

**Theorem 2.7** *Let $T$ be a tree of order $n \geq 4$. Then*

$$W(\overline{M}(T)) = 2n(n-1) + \frac{1}{2}\sum_{i=1}^{n} d_i^2.$$

*Proof.* Let $T$ be a tree with $n$ vertices and $m$ edges. Then $\overline{M}(T)$ has $n_1$ vertices and $m_1$ edges.

For $n \geq 4$, in $\overline{M}(T)$ the distance between adjacent vertices is one and the distance between nonadjacent vertices is two.

Therefore $diam(\overline{M}(T)) = 2$.

From Theorem 1.1, $W(\overline{M}(T)) = n_1(n_1 - 1) - m_1$. (4)

But $n_1 = n + m = 2n - 1$ (5)

$$m_1 = \binom{n+m}{2} - (m + \frac{1}{2}\sum_{i=1}^{n} d_i^2)$$

$$m_1 = 2(n-1)^2 - \frac{1}{2}\sum_{i=1}^{n} d_i^2. \quad (6)$$

From equations (4), (5) and (6)

$$W(\overline{M}(T)) = (2n-1)(2n-2) - 2(n-1)^2 + \frac{1}{2}\sum_{i=1}^{n} d_i^2$$

$$W(\overline{M}(T)) = 2n(n-1) + \frac{1}{2}\sum_{i=1}^{n} d_i^2.$$

**Corollary 2.8** *If $P_n$ is a path of order $n \geq 4$, then*

$$W(\overline{M}(P_n)) = 2n^2 - 3.$$

*Proof.* From Theorem 2.7,

$$W(\overline{M}(T)) = 2n(n-1) + \frac{1}{2}\sum_{i=1}^{n} d_i^2.$$

In case of path, two pendant vertices have degree $1$ and remaining $n-2$ vertices have degree 2. Therefore

$$\sum_{i=1}^{n} d_i^2 = 2 + 4(n-2)$$

$$W(\overline{M}(P_n)) = 2n(n-1) + \frac{1}{2}[2 + 4(n-2)]$$

$$W(\overline{M}(P_n)) = 2n^2 - 3.$$

**Corollary 2.9** *If $S_n$ is a star graph of order $n \geq 4$ then,*

$$W(\overline{M}(S_n)) = \frac{5n(n-1)}{2}.$$

*Proof.* From Theorem 2.7,

$$W(\overline{M}(T)) = 2n(n-1) + \frac{1}{2}\sum_{i=1}^{n} d_i^2.$$

In case of star graph, one vertex has degree $n-1$ and remaining $n-1$ vertices have degree 1.

Therefore $\sum_{i-1}^{n} d_i^2 = (n-1)^2 + n - 1 = n^2 - n$

$$W(\overline{M}(S_n)) = 2n(n-1) + \frac{1}{2}(n^2 - n)$$

$$W(\overline{M}(S_n)) = \frac{5n(n-1)}{2}.$$

**Corollary 2.10**

*For any tree $T$ of order $n$,*

$$W(\overline{M}(P_n)) \le W(\overline{M}(T)) \le W(\overline{M}(S_n)).$$

*Proof.* From Theorem 2.7,

$$W(\overline{M}(T)) = 2n(n-1) + \frac{1}{2}\sum_{i=1}^{n} d_i^2 .$$

In case of tree $\sum_{i=1}^{n} d_i^2$ is maximum for star and minimum for path.

Therefore $W(\overline{M}(P_n)) \le W(\overline{M}(T)) \le W(\overline{M}(S_n))$.

**Theorem 2.11** If $K_n$ *is a complete graph of order* $n \ge 4$, *then*

$$W(\overline{M}(K_n)) = \frac{n(n^3 + 6n^2 - 5n - 2)}{8}.$$

*Proof.* Let $K_n$ be a complete graph with $n \ge 4$ vertices and $m$ edges. Then $\overline{M}(K_n)$ has $n_1$ vertices and $m_1$ edges.

For $n \ge 4$, in $\overline{M}(K_n)$ the distance between adjacent vertices is one and the distance between nonadjacent vertices is two, therefore $diam(\overline{M}(K_n)) = 2$.

From Theorem 1.1, $W(\overline{M}(K_n)) = n_1(n_1 - 1) - m_1$ (7)

But $n_1 = n + m = n + \frac{n(n-1)}{2} = \frac{n^2+n}{2}$ (8)

$$m_1 = \binom{n+m}{2} - (m + \frac{1}{2}\sum_{i=1}^{n} d_i^2) = \binom{\frac{n^2+n}{2}}{2} - (\frac{n(n-1)}{2} + \frac{1}{2}\sum_{i=1}^{n} d_i^2).$$

From Theorem 1.6, $\sum_{i=1}^{n} d_i^2 = n(n-1)^2$

$$m_1 = \binom{\frac{n^2+n}{2}}{2} - (\frac{n(n-1)}{2} + \frac{1}{2}n(n-1)^2) = \frac{n^4 - 2n^3 + 3n^2 - 2n}{8}. \quad (9)$$

From equations (7), (8) and (9)

$$W(\overline{M}(K_n)) = (\frac{n^2+n}{2})(\frac{n^2+n}{2} - 1) - (\frac{n^4 - 2n^3 + 3n^2 - 2n}{8})$$

$$W(\overline{M}(K_n)) = \frac{n(n^3 + 6n^2 - 5n - 2)}{8}.$$

**Theorem 2.12** If $C_n$ *is a cycle of order* $n \ge 4$, *then*

$$W(\overline{M}(C_n)) = 2n(n+1).$$

*Proof.* Let $C_n$ be a cycle with $n$ vertices and $m$ edges. Then $\overline{M}(C_n)$ has $n_1$ vertices and $m_1$ edges. For $n \ge 4$ in $\overline{M}(C_n)$ the distance between adjacent vertices is one and the distance between nonadjacent vertices is two, therefore $diam\overline{M}(C_n) = 2$.

From Theorem 1.1, $W(\overline{M}(C_n)) = n_1(n_1 - 1) - m_1$ (10)

But $n_1 = n + m = 2n$ (11)

$$m_1 = \binom{n+m}{2} - (m + \frac{1}{2}\sum_{i-1}^{n} d_i^2) = \binom{2n}{2} - (n + \frac{1}{2}\sum_{i=1}^{n} d_i^2).$$

In case of cycle degree of each vertex is $2$.

Therefore $\sum_{i=1}^{n} d_i^2 = 4n$

$$m_1 = \binom{2n}{2} - (n + \frac{4n}{2}) = 2n(n-2). \quad (12)$$

From equations (10), (11) and (12)

$$W(\overline{M}(C_n)) = 2n(2n-1) - 2n(n-2)$$

$$W(\overline{M}(C_n)) = 2n(n+1).$$

**Theorem 2.13** If $G$ *is a connected graph of order* $n$, *then*

$$W(G) < W(M(G)).$$

*Proof.* If $G$ is graph with $n$ vertices and $m$ edges then $M(G)$ is a middle graph of $G$ with $n+m$ vertices and $m + \frac{1}{2}\sum_{i=1}^{n} d_i^2$ edges.

Wiener index of graph increases when new vertices are added to the graph $G$.

Therefore $W(G) < W(M(G))$.

**Lemma 2.14** If $G$ *is connected graph of order n and size m, then*

$$4(n-1)^2 \le W(M(G)) \le \frac{n(n^3+n-2)}{4}.$$

Upper bound attains if $G$ is a complete graph and lower bound attains if $G$ is a star graph.

*Proof.* The middle graph $M(G)$ of G has $n+m$ vertices and $m+\frac{1}{2}\sum_{i=1}^{n} d_i^2$ edges [15]

$G$ has maximum edges if and only if $G \cong K_n$, $M(G)$ has maximum number of vertices if and only if $G \cong K_n$.

Wiener index of a graph increases when new vertices are added to the graph and $M(K_n)$ has maximum number of vertices compared with any other $M(G)$.

Therefore $W(M(G)) \le W(M(K_n))$.

From Theorem 2.5, $W(M(K_n)) = \frac{n(n^3+n-2)}{4}$.

Therefore $W(M(G)) \le \frac{n(n^3+n-2)}{4}$ with equality if and only if $G \cong K_n$. (13)

Any graph $G$ has minimum edges if and only if $G \cong T$ and $M(G)$ has minimum number of vertices if and only if $G \cong T$.

Wiener index of a graph increases when new vertices are added to the graph and $M(T)$ has minimum number of vertices compared with any other $M(G)$.

Therefore $W(M(T)) \le W(M(G))$. (14)

From Corollary 2.4, $W(M(S_n)) \le W(M(T))$. (15)

From equations (14) and (15)

$W(M(S_n)) \le W(M(G))$.

Since $W(M(S_n)) = 4(n-1)^2$, it follows that $4(n-1)^2 \le W(M(G))$ with equality if and only if $G \cong S_n$. (16)

From equations (13) and (16),

$$4(n-1)^2 \le W(M(G)) \le \frac{n(n^3+n-2)}{4}.$$

**Lemma 2.15** *For any connected graph* $G$ *of order* $n \ge 4$ *vertices,*

$$2n^2 - 3 \le W(\overline{M}(G)) \le \frac{n(n^3+6n^2-5n-2)}{8}.$$

Upper bound attains if $G$ is a complete graph and lower bound attains if $G$ is a path.

*Proof.* Let $G$ be connected graph with $n \ge 4$ vertices and $m$ edges. Then $M(G)$ has $n+m$ vertices and $m+\frac{1}{2}\sum_{i=1}^{n} d_i^2$ edges.

$\overline{M}(K_n)$ has $n+m$ vertices and $\binom{n+m}{2} - (m+\frac{1}{2}\sum_{i=1}^{n} d_i^2)$ edges.

$G$ has maximum edges if and only if $G \cong K_n$, $\overline{M}(G)$ has maximum number of vertices if and only if $G \cong K_n$.

Wiener index of a graph increases when new vertices are added to the graph and $\overline{M}(K_n)$ has maximum number of vertices compared to any other $\overline{M}(G)$.

Therefore $W(\overline{M}(G)) \le W(\overline{M}(K_n))$.

From Theorem 2.11,

$$W(\overline{M}(K_n)) = \frac{n(n^3+6n^2-5n-2)}{8}.$$

Therefore $W(\overline{M}(G)) \le \frac{n(n^3+6n^2-5n-2)}{8}$. (17)

For any connected graph $G$ with $n \ge 4$ vertices, $G$ has minimum number of vertices if and only if $G \cong T$.

Wiener index of a graph increases when new vertices are added to a graph and $\overline{M}(T)$ has minimum number of vertices compared to any other $\overline{M}(G)$.

Therefore $W(\overline{M}(T)) \leq W(\overline{M}(G))$. (18)

From Theorem 2.7,

$W(\overline{M}(T)) = 2n(n-1) + \frac{1}{2}\sum_{i=1}^{n} d_i^2$ and in case of tree $\sum_{i=1}^{n} d_i^2$ is maximum for star and minimum for path.

Therefore $W(\overline{M}(P_n)) \leq W(\overline{M}(T))$. (19)

From equations (18) and (19)

$W(\overline{M}(P_n)) \leq W(\overline{M}(T)) \leq W(\overline{M}(G))$.

Therefore $W(\overline{M}(P_n)) \leq W(\overline{M}(G))$.

From Corollary 2.8, $W(\overline{M}(P_n)) = 2n^2 - 3$.

Therefore $2n^2 - 3 \leq W(\overline{M}(G))$. (20)

From equations (17) and (20)

$$2n^2 - 3 \leq W(\overline{M}(G)) \leq \frac{n(n^3 + 6n^2 - 5n - 2)}{8}.$$

The following theorem gives the Nordhaus-Gaddum type inequality for Wiener index of middle graph.

**Theorem 2.16** *For any graph $G$ with $n \geq 4$ vertices,*

$$6n^2 - 8n + 1 \leq W(M(G)) + W(\overline{M}(G)) \leq \frac{3n(n^3 + 2n^2 - n - 2)}{8}.$$

*Proof.* From Lemmas 2.14 and 2.15, we have

$$4(n-1)^2 + 2n^2 - 3 \leq W(M(G)) + W(\overline{M}(G)) \leq \frac{n^4 + n^2 - 2n}{4} + \frac{n^4 + 6n^3 - 5n^2 - 2n}{8}.$$

Thus

$$6n^2 - 8n + 1 \leq W(M(G)) + W(\overline{M}(G)) \leq \frac{3n(n^3 + 2n^2 - n - 2)}{8}$$


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