



NON-NEIGHBOUR IRREGULAR DERIVED GRAPHS

B. BASAVANAGOUD^{1*} AND VEENA R. DESAI¹

¹Department of Mathematics, Karnatak University, Dharwad - 580 003, Karnataka, India.

AUTHORS' CONTRIBUTIONS

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Received: 4th August 2016

Accepted: 22nd August 2016

Published: 27th August 2016

Original Research Article

ABSTRACT

A graph G is said to be non-neighbour irregular graph if no two nonadjacent vertices of G have same degree. In this paper, we obtain non-neighbour irregular derived graphs such as complement graphs, line graphs, jump graphs, subdivision graphs, paraline graphs, semitotal-point graphs, semitotal-line graphs, total graphs, quasi-total graphs and quasivertex-total graphs.

Keywords: Degree; non-neighbour irregular graph; derived graphs.

2010 mathematics subject classification: 05C07.

1 Introduction

Throughout this paper, we consider only undirected, finite and simple graphs. Let G be such graph with vertex set $V(G)$. The *degree of a vertex* $v \in V(G)$ is the number of vertices adjacent to v and is denoted by $d_G(v)$. The *neighbourhood of a vertex* $u \in V(G)$ is the set of vertices which are adjacent to u and is denoted by $N(u)$. Let G be a graph with n vertices and m edges. Notations and terminology that we do not define here can be found in [1,2].

A graph G is said to be *regular* if all its vertices have the same degree. A connected graph G is said to be *highly irregular* [3] if each neighbour of any vertex has different degree. It is called *k -neighbourhood regular graph* [4] if each of its vertex is adjacent to exactly k -vertices of the same degree. The graph G is said to be *neighbourly irregular graph* [5], abbreviated as NI graph, if no two adjacent vertices of G have the same degree. The graph G is said to be *non-neighbour irregular* if no two nonadjacent vertices of G have the same degree. This concept was introduced in [6] and constructed NNI graphs of order $n - k + 1$ and n for a given n and a partition of n with k distinct parts and proved some properties of NNI graphs related to clique graph, vertex covering number, edge covering number, vertex independence number, edge independence number and domination number. The Fig. 1. depicts an examples of NNI graphs.

*Corresponding author: Email: b.basavanagoud@gmail.com;

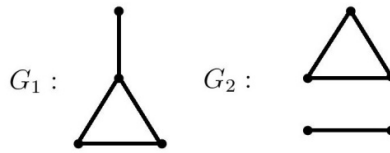


Fig. 1. NNI graphs

2 Derived Graphs

In this paper we considered the following graphs derived from the parent graph G :

1. The **complement** of G , denoted by \bar{G} , is a graph which has the same vertex set as G , in which two vertices are adjacent if and only if they are nonadjacent in G and $d_{\bar{G}}(v) = n - 1 - d_G(v)$ holds for all $v \in V(G)$.
2. The **line graph** $L(G)$ of G is the graph [7] whose vertex set is $E(G)$ in which two vertices are adjacent if and only if they are adjacent in G . If $e = uv$ is an edge of G then $d_{L(G)}(e) = d_G(u) + d_G(v) - 2$.
3. The **jump graph** $J(G)$ of G is the graph [8] whose vertex set is $E(G)$ in which two vertices are adjacent if and only if they are nonadjacent in G . Readers interested in more information on jump graph can be referred to [8,9,10].
4. The **subdivision graph** $S(G)$ of a graph G [1] is obtained from G by inserting a new vertex into every edge of G .
5. The **paraline graph** $PL(G)$ is a line graph of subdivision graph of G .
6. The **semitotal-point graph** $T_2(G)$ as the graph [11] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if (i) they are adjacent vertices of G or (ii) one is a vertex of G and other is an edge of G incident with it. If u is a vertex of G , then $d_{T_2(G)}(u) = 2d_G(u)$. If e is an edge of G , then $d_{T_2(G)}(e) = 2$.
7. The **semitotal-line graph** $T_1(G)$ as the graph [11] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if (i) they are adjacent edges of G or (ii) one is a vertex of G and other is an edge of G incident with it. If u is a vertex of G , then $d_{T_1(G)}(u) = d_G(u)$. If $e = uv$ is an edge of G , then $d_{T_1(G)}(e) = d_G(u) + d_G(v)$.
8. The **total graph** $T(G)$ as the graph [1] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if (i) they are adjacent vertices of G or (ii) they are adjacent edges of G or (iii) one is a vertex of G and other is an edge of G incident with it. If u is a vertex of G , then $d_{T(G)}(u) = 2d_G(u)$. If $e = uv$ is an edge of G , then $d_{T(G)}(e) = d_G(u) + d_G(v)$.
9. The **quasi-total graph** $P(G)$ as the graph [12] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if (i) they are nonadjacent vertices of G or (ii) they are adjacent edges of G or (iii) one is a vertex of G and other is an edge of G incident with it. If u is a vertex of G , then $d_{P(G)}(u) = n - 1$. If $e = uv$ is an edge of G , then $d_{P(G)}(e) = d_G(u) + d_G(v)$. Readers interested in more information on quasi-total graph can be referred to [12,13,14].
10. The **quasivertex-total graph** $Q(G)$ as the graph [15] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if (i) they are adjacent vertices of G or (ii) they are nonadjacent vertices of G (iii) they are adjacent edges of G or (iv) one is a vertex of G and other is an edge of G incident with it. If u is a vertex of G , then $d_{Q(G)}(u) = n - 1 + d_G(u)$. If $e = uv$ is an edge of G , then $d_{Q(G)}(e) = d_G(u) + d_G(v)$.

In Fig. 2 self-explanatory examples of these derived graphs are depicted.

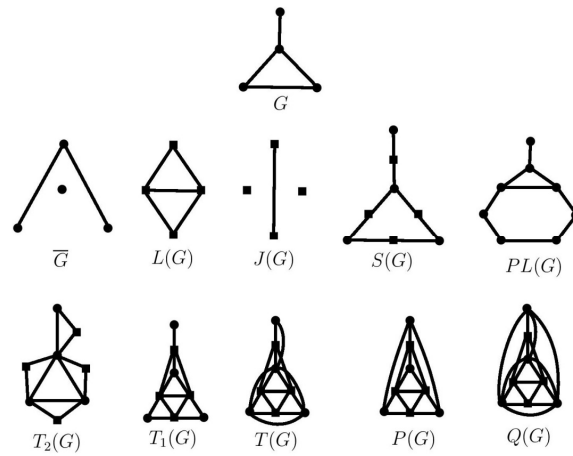


Fig. 2. Graph G and its derived graphs

Fig. 2 various graphs derived from the graph G . The vertices of these derived graphs (except the paraline graph PL), corresponding to the vertices of the parent graph G , are indicated by circles. The vertices of these graphs corresponding to the edges of the parent graph G are indicated by squares.

Theorem 2.1. [16] For any graph G , its line graph $L(G)$ is NI graph if and only if $N(u)$ contains all vertices of different degree for all $u \in V(G)$.

Theorem 2.2. [6] If a graph G is NNI graph, then \bar{G} is not NNI graph.

In this paper we obtain non-neighbour irregular derived graphs such as complement graphs, line graphs, jump graphs, subdivision graphs, paraline graphs, semitotal-point graphs, semitotal-line graphs, total graphs, quasi-total graphs and quasivertex-total graphs.

3 Results

Theorem 3.1. For any graph G , the complement graph \bar{G} is NNI graph if and only if G is NI graph.

Proof. Let \bar{G} be NNI graph. To prove G is NI graph, i. e. to prove every pair of adjacent vertices in G have different degree, on the contrary, suppose two adjacent vertices u and v of G have the same degree, that is, $d_G(u) = d_G(v)$. Therefore $n - 1 - d_G(u) = n - 1 - d_G(v)$.

This implies that, $d_{\bar{G}}(u) = d_{\bar{G}}(v)$. Thus, \bar{G} is not NNI graph, a contradiction. Hence G is NI graph.

Conversely, Let G be NI graph. Therefore for every pair of adjacent vertices u and v of G , $d_G(u) \neq d_G(v)$. Therefore $n - 1 - d_G(u) \neq n - 1 - d_G(v)$. This implies that, $d_{\bar{G}}(u) \neq d_{\bar{G}}(v)$. Thus, \bar{G} is NNI graph.

Theorem 3.2. For any graph G , the line graph $L(G)$ is NNI graph if and only if $V(G) \setminus N(u)$ contains all vertices of different degree for all $u \in V(G)$.

Proof. Let $L(G)$ be NNI graph. To prove that $V(G) \setminus N(u)$ contains all vertices of different degree for all $u \in V(G)$, on the contrary, suppose $V(G) \setminus N(u)$ contains two vertices v and w such that $d_G(v) = d_G(w)$, where $v, w \in V(G) \setminus N(u)$. Then there exists two nonadjacent vertices $e_1 = uv$ and $e_2 = uw$ in $L(G)$ such that $d_{L(G)}(e_1) = d_{L(G)}(e_2)$. Thus, $L(G)$ is not NNI graph, a contradiction. Hence $V(G) \setminus N(u)$ contains all vertices of different degree.

Conversely, $d_G(v) \neq d_G(w)$, for all $v, w \in V(G) \setminus N(u)$ and $u \in V(G)$. Therefore $d_G(u) + d_G(v) - 2 \neq d_G(u) + d_G(w) - 2$. That is, $d_{L(G)}(e_1) = d_{L(G)}(e_2)$, where $e_1 = uv$ and $e_2 = uw$ are nonadjacent in $L(G)$. Hence $L(G)$ is NNI graph.

From Theorems 2.2 and 3.2, we have the following corollary.

Corollary 3.3. *If the line graph $L(G)$ is NNI graph, then the jump graph $J(G)$ is not NNI graph.*

From Theorems 2.1 and 3.1, we have the following corollary.

Corollary 3.4. *For any graph G , the jump graph $J(G)$ is NNI graph if and only if $L(G)$ is NI graph.*

Theorem 3.5. *For any graph G , the subdivision graph $S(G)$ is not NNI graph.*

Proof. We prove the result by considering the following cases:

Case 1. If $G = K_2$, then $S(G) = P_3$ which is not NNI graph.

Case 2. If $G \neq K_2$, then there exists at least two nonadjacent vertices $e_i = v_r v_s$ and $e_j = v_s v_w$ in $S(G)$ with $d_{S(G)}(e_1) = d_{S(G)}(e_2) = 2$. Therefore $S(G)$ is not NNI graph.

Theorem 3.6. *For any graph $G \neq K_2$, the paraline graph $PL(G)$ is not NNI graph.*

Proof. Since $G \neq K_2$, then there exists at least two nonadjacent vertices u and v in $PL(G)$ with $d_{PL(G)}(u) = d_{PL(G)}(v)$. Therefore $PL(G)$ is not NNI graph.

Theorem 3.7. *For any graph $G \neq K_2$, the semitotal-point graph $T_2(G)$ is not NNI graph.*

Proof. Since $G \neq K_2$, then there exists at least two nonadjacent vertices $e_i = v_r v_s$ and $e_j = v_s v_w$ in $T_2(G)$ with $d_{T_2(G)}(e_1) = d_{T_2(G)}(e_2) = 2$. Therefore $T_2(G)$ is not NNI graph.

Theorem 3.8. *For any graph G , the semitotal-line graph $T_1(G)$ is NNI graph if and only if all vertices of G have different degree, $L(G)$ is NNI graph and $d_G(w_i) \neq d_G(u_i) + d_G(v_i)$, $\forall w_i \in V(G)$ and $e_i = u_i v_i \in E(G)$.*

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ be the vertex set and $E(G) = \{e_1, e_2, \dots, e_m\}$ be the edge set of G . Suppose all vertices of G have different degree, $L(G)$ is NNI graph and $d_G(w_i) \neq d_G(u_i) + d_G(v_i)$, $\forall w_i \in V(G)$ and $e_i = u_i v_i \in E(G)$. In $T_1(G)$ the vertices x and y are nonadjacent, then $x, y \in V(G)$ or $x, y \in E(G)$, where x and y are nonadjacent in G or $x \in V(G)$ and $y \in E(G)$, where x and y are nonincident in G .

- (a) $x, y \in V(G)$. Since $d_G(x) \neq d_G(y)$, $d_{T_1(G)}(x) = d_G(x) \neq d_G(y) = d_{T_1(G)}(y)$.
- (b) $x, y \in E(G)$, where x and y are nonadjacent in G . Let $x = v_i v_j$ and $y = v_k v_w$, so that x and y are nonadjacent in $T_1(G)$. Therefore $d_G(v_i) + d_G(v_j) - 2 \neq d_G(v_k) + d_G(v_w) - 2$, as $L(G)$ is NNI graph. This implies that $d_{T_1(G)}(x) \neq d_{T_1(G)}(y)$.
- (c) $x \in V(G)$ and $y \in E(G)$, where x and y are nonincident in G . Let $e_i = u_i v_i \in E(G)$ and $w_i \in V(G)$.

$$\begin{aligned} d_{T_1(G)}(e_i) &= d_G(u_i) + d_G(v_i) \\ &\neq d_G(w_i), \text{ as } d_G(w_i) \neq d_G(u_i) + d_G(v_i) \\ &\neq d_{T_1(G)}(w_i). \end{aligned}$$

Thus in all cases $T_1(G)$ is NNI graph.

Conversely, suppose $T_1(G)$ is NNI graph. We have to prove that all vertices of G have different degree, $L(G)$ is NNI graph and $d_G(w_i) \neq d_G(u_i) + d_G(v_i)$, $\forall w_i \in V(G)$ and $e_i = u_i v_i \in E(G)$. If all vertices of G does not have same degree in G , then there exists at least two nonadjacent vertices v_i and u_i in $T_1(G)$ such that $d_{T_1(G)}(v_i) = d_{T_1(G)}(u_i)$. A contradiction to $T_1(G)$ is NNI graph.

Suppose $L(G)$ is not NNI graph, then there exists two nonadjacent vertices $e_i = v_r v_s$ and $e_j = v_k v_w$ in $L(G)$ with $d_{L(G)}(e_i) = d_{L(G)}(e_j)$. Therefore $d_G(v_r) + d_G(v_s) = d_G(v_k) + d_G(v_w)$. This implies that, $d_{T_1(G)}(e_i) = d_{T_1(G)}(e_j)$. A contradiction to $T_1(G)$ is NNI graph.

Suppose $d_G(w_i) = d_G(u_i) + d_G(v_i)$, for some $w_i \in V(G)$ and $e_i = u_i v_i \in E(G)$. Let $e_i = u_i v_i$ be an edge of G . Then $d_{T_1(G)}(e_i) = d_G(u_i) + d_G(v_i) = d_G(w_i) = d_{T_1(G)}(w_i)$. Again a contradiction to $T_1(G)$ is NNI graph.

Theorem 3.9. For any graph G , the total graph $T(G)$ is NNI graph if and only if G is NNI graph, $L(G)$ is NNI graph and $2d_G(w_i) \neq d_G(u_i) + d_G(v_i)$, $\forall w_i \in V(G)$ and $e_i = u_i v_i \in E(G)$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ be the vertex set and $E(G) = \{e_1, e_2, \dots, e_m\}$ be the edge set of G . Suppose G is NNI graph, $L(G)$ is NNI graph and $2d_G(w_i) \neq d_G(u_i) + d_G(v_i)$, $\forall w_i \in V(G)$ and $e_i = u_i v_i \in E(G)$. In $T(G)$ the vertices x and y are nonadjacent, then $x, y \in V(G)$, where x and y are nonadjacent in G or $x, y \in E(G)$, where x and y are nonadjacent in G or $x \in V(G)$ and $y \in E(G)$, where x and y are nonincident in G .

- (a) $x, y \in V(G)$, where x and y are nonadjacent in G . Since $d_G(x) \neq d_G(y)$, $d_{T(G)}(x) = 2d_G(x) \neq 2d_G(y) = d_{T(G)}(y)$.
- (b) $x, y \in E(G)$, where x and y are nonadjacent in G . Let $x = v_i v_j$ and $y = v_k v_w$, so that x and y are nonadjacent in $T(G)$. Therefore $d_G(v_i) + d_G(v_j) - 2 \neq d_G(v_k) + d_G(v_w) - 2$, as $L(G)$ is NNI graph. This implies that, $d_{T(G)}(x) \neq d_{T(G)}(y)$.
- (c) $x \in V(G)$ and $y \in E(G)$, where x and y are nonincident in G . Let $e_i = u_i v_i \in E(G)$ and $w_i \in V(G)$.

$$\begin{aligned} d_{T(G)}(e_i) &= d_G(u_i) + d_G(v_i) \\ &\neq d_G(w_i), \text{ as } 2d_G(w_i) \neq d_G(u_i) + d_G(v_i) \\ &\neq d_{T(G)}(w_i). \end{aligned}$$

Thus in all cases $T(G)$ is NNI graph.

Conversely, suppose $T(G)$ is NNI graph. We have to prove that G is NNI graph, $L(G)$ is NNI graph and $2d_G(w_i) \neq d_G(u_i) + d_G(v_i)$, $\forall w_i \in V(G)$ and $e_i = u_i v_i \in E(G)$. If G is not NNI graph, then there exists at least two nonadjacent vertices u_i and v_i , such that $d_G(u_i) = d_G(v_i)$. This implies $d_{T(G)}(u_i) = d_{T(G)}(v_i)$. A contradiction to $T(G)$ is NNI graph.

Suppose $L(G)$ is not NNI graph, then there exists two nonadjacent vertices $e_i = v_r v_s$ and $e_j = v_k v_w$ in $L(G)$ with $d_{L(G)}(e_i) = d_{L(G)}(e_j)$. Therefore $d_G(v_r) + d_G(v_s) = d_G(v_k) + d_G(v_w)$. This implies that, $d_{T(G)}(e_i) = d_{T(G)}(e_j)$. A contradiction to $T(G)$ is NNI graph.

Suppose $2d_G(w_i) = d_G(u_i) + d_G(v_i)$, for some $w_i \in V(G)$ and $e_i = u_i v_i \in E(G)$. Let $e_i = u_i v_i$ be an edge of G . Then $d_{T(G)}(e_i) = d_G(u_i) + d_G(v_i) = 2d_G(w_i) = d_{T(G)}(w_i)$. Again a contradiction to $T(G)$ is NNI graph.

Theorem 3.10. For any graph G , the quasi-total graph $P(G)$ is not NNI graph.

Proof. We prove the result by considering the following cases:

Case 1. If $G = K_2$, then $P(G) = P_3$ which is not NNI graph.

Case 2. If $G \neq K_2$, then there at least exists two nonadjacent vertices v_i and u_i in $P(G)$ with $d_{P(G)}(v_i) = d_{P(G)}(u_i) = n - 1$. Therefore $P(G)$ is not NNI graph.

Theorem 3.11. For any graph G , the quasivertex-total graph $Q(G)$ is NNI graph if and only if $L(G)$ is NNI graph and $n - 1 + d_G(w_i) \neq d_G(u_i) + d_G(v_i)$, $\forall w_i \in V(G)$ and $e_i = u_i v_i \in E(G)$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ be the vertex set and $E(G) = \{e_1, e_2, \dots, e_m\}$ be the edge set of G . Suppose $L(G)$ is NNI graph and $d_G(w_i) = n - 1 + d_G(w_i) \neq d_G(u_i) + d_G(v_i)$, $\forall w_i \in V(G)$ and $e_i = u_i v_i \in E(G)$. In $Q(G)$ the vertices x and y are nonadjacent, then $x, y \in E(G)$, where x and y are nonadjacent in G or $x \in V(G)$ and $y \in E(G)$, where x and y are nonincident in G .

(a) $x, y \in E(G)$, where x and y are nonadjacent in G . Let $x = v_i v_j$ and $y = v_k v_w$, so that x and y are nonadjacent in $Q(G)$. Therefore $d_G(v_i) + d_G(v_j) - 2 \neq d_G(v_k) + d_G(v_w) - 2$ as $L(G)$ is NNI graph.

This implies that, $d_{Q(G)}(x) \neq d_{Q(G)}(y)$.

(b) $x \in V(G)$ and $y \in E(G)$, where x and y are nonincident in G . Let $e_i = u_i v_i \in E(G)$ and $w_i \in V(G)$.

$$\begin{aligned} d_{P(G)}(e_i) &= d_G(u_i) + d_G(v_i) \\ &\neq n - 1 + d_G(w_i), \text{ as } n - 1 + d_G(w_i) \neq d_G(u_i) + d_G(v_i) \\ &\neq d_{Q(G)}(w_i). \end{aligned}$$

Thus in all cases $Q(G)$ is NNI graph.

Conversely, suppose $Q(G)$ is NNI graph. We have to prove that $L(G)$ is NNI graph and $n - 1 + d_G(w_i) \neq d_G(u_i) + d_G(v_i)$, $\forall w_i \in V(G)$ and $e_i = u_i v_i \in E(G)$. Suppose $L(G)$ is not NNI graph, then there exists two nonadjacent vertices $e_i = v_r v_s$ and $e_j = v_k v_w$ in $L(G)$ with $d_{L(G)}(e_i) = d_{L(G)}(e_j)$. Therefore $d_G(v_r) + d_G(v_s) = d_G(v_k) + d_G(v_w)$. This implies that, $d_{Q(G)}(e_i) = d_{Q(G)}(e_j)$. A contradiction to $Q(G)$ is NNI graph.

Suppose $n - 1 + d_G(w_i) = d_G(u_i) + d_G(v_i)$, for some $w_i \in V(G)$ and $e_i = u_i v_i \in E(G)$. Let $e_i = u_i v_i$ be an edge of G . Then $d_{Q(G)}(e_i) = d_G(u_i) + d_G(v_i) = n - 1 + d_G(w_i) = d_{Q(G)}(w_i)$. Again a contradiction to $Q(G)$ is NNI graph.

4 Conclusion

In this paper, we characterize the non-neighbour irregular derived graphs such as complement graphs, line graphs, jump graphs, semitotal-line graphs, total graphs, and quasivertex-total graphs. In addition we have shown that for any graph G , subdivision graphs and quasi-total graphs are not non-neighbour irregular graphs. Further we proved that for any graph $G \neq K_2$, paraline graphs and semitotal-point graphs are not non-neighbour irregular graphs. These results can be extended for any other graph valued functions.

Acknowledgement

This research is supported by UGC-SAP DRS-III, New Delhi, India for 2016-2021: F.510/3/DRS-III/2016 (SAP-I) Dated : 29th Feb. 2016.

This research is supported by UGC-National Fellowship (NF) New Delhi. No. F./2014-15/NFO-2014-15-OBC-KAR-25873/(SA-III/Website) Dated: March-2015.

Competing Interests

Authors have declared that no competing interests exist.

References

- [1] Harary F. Graph theory. Addison-Wesley, Reading, Mass; 1969.
- [2] Kulli VR. College graph theory. Vishwa International Publications, Gulbarga, India; 2012.
- [3] Alavi Y, Chartrand G, Chang FRK, Erdos P, Graham HL, Oellermann OR. Highly irregular graphs. J. Graph Theory. 1987;11:235-249.
- [4] Balakrishnan R, Selvam A. K-neighbourhood regular graphs. Proc. Natl. Semin. Graph Theory. 1996;35-45.
- [5] Bhraagsam SG, Ayyaswamy SK. Neighbourly irregular graphs. Indian J. Pure Appl. Math. 2004;35(3): 389-399.
- [6] Basavanagoud B, Ramane HS, Desai VR. Non-neighbour irregular graphs. Bull. Math. Sci. Appl. 2016;15:1-7.
- [7] Whitney H. Congruent graphs and the connectivity of graphs. Amer. J. Math. 1932;54:150-168.
- [8] Chartrand G, Hevia H, Jarette EB, Schultz M. Subgraph distance in graphs defined by edge transfers. Discrete Math. 1997;170:63-79.
- [9] Hevia H, VanderJagt DW, Zhang P. On the planarity of jump graphs. Discrete Math. 2000;220: 119-129.
- [10] Wu B, Meng J. Hamiltonian jump graphs. Discrete Math. 2004;289:95-106.
- [11] Sampathkumar E, Chikkodimath SB. Semitotal graphs of a graph-I. J. Karnatak Univ. Sci. 1973;18:274-280.
- [12] Sastry DVS, Syam Prasad Raju B. Graph equations for line graphs, total graphs, middle graphs and quasi-total graphs. Discrete Math. 1984;48:113-119.
- [13] Basavanagoud B. Quasi-total graphs with crossing numbers. J. Discrete Math. Sci. Cryptogr. 1998;1:133-142.
- [14] Kulli VR, Basavanagoud B. Traversability and planarity of quasi-total graphs. Bull. Cal. Math. Soc. 2002;94(1):1-6.
- [15] Kulli VR, Basavanagoud B. On the quasivertex-total graph of a graph. J. Karnatak Univ. Sci. 1998;42:1-7.
- [16] Walikar HB, Halkarni SB, Ramane HS, Tavakoli M, Ashrafi AR. On neighbourly irregular graphs. Kragujevac J. Math. 2015;39(1):31-39.