Non-neighbour irregular graphs

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Abstract: A graph G is said to be non-neighbour irregular graph if no two nonadjacent vertices of G have same degree. This paper suggests the methods of construction of non-neighbour irregular graphs. This paper also includes a few properties possessed by these non-neighbour irregular graphs.

1 Introduction

Throughout this paper we consider only undirected, finite and simple graphs. Let G be such graph with vertex set $V(G)$. The *degree of a vertex* $v \in V(G)$ is the number of vertices adjacent to v and is denoted by $d_G(v)$. Let $K_{p,q}$ is a complete bipartite graph on $p + q$ vertices and P_n is a path with n vertices. If G and H are graphs with the property that the *identification* of any vertices of G with an arbitrary vertex of H results in a unique graph (up to isomorphism), then we write $G \cdot H$ for this graph. Let $\lceil x \rceil (\lfloor x \rfloor)$ denote the least (greatest) integer greater (less) than or equal to x. Defining a new graph from a given graph, by using incident relationship between vertices and edges and adjacency relation between two vertices or two edges is known as *graph transformation*. Sometimes graph transformation is referred as a graph valued function. Perhaps one of the first graph transformation of a graph G is its *complement* denoted by $\overline{G}$, which is a graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are nonadjacent in G and $d_{\overline{G}}(v) = n - 1 - d_G(v)$ holds for all $v \in V(G)$, where n is the number of vertices of G. A *self-complementary* graph is isomorphic with its complement. Notations and terminology that we do not define here can be found in [4, 5].

Regular graphs are those graphs for which each vertex has same degree. There are plenty of regular graphs, for example, complete graph. The problem arises when a graph is not regular. If it is irregular how much of irregularity is thrust upon its vertices? In this connection three new concepts called highly irregular graphs, k-neighbourhood graphs and neighbourly irregular graphs [1, 2, 3] have evolved. A connected graph G is said to be *highly irregular* if each neighbour of any vertex has different degree. A connected graph G is said to be a *k-neighbourhood regular* graph if each of its vertices is adjacent to exactly k-vertices of the same degree. A connected graph G is said to be *neighbourly irregular* graph abbreviated as NI graph if no two adjacent vertices of G have the same degree.

Inspired by these three definitions we define the concept of non-neighbour irregular graphs abbreviated as NNI graphs.

2 Non-neighbour Irregular graphs

A graph G is said to be *non-neighbour irregular* if no two nonadjacent vertices of G have the same degree. For example, the graphs shown in Fig. 1 are NNI graphs.
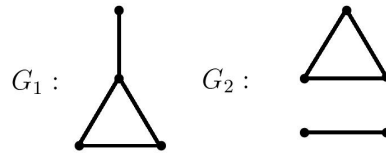
Figure 1.

This concept is helpful to study the NI for the graph transformation $G \rightarrow \overline{G}$.

Note that the graph which is not NI is not necessarily a NNI graph. For example consider the graph G_1 shown in Fig. 2 is not NI, but G_1 is NNI. The graph G_2 shown in Fig. 2 is neither NI nor NNI.
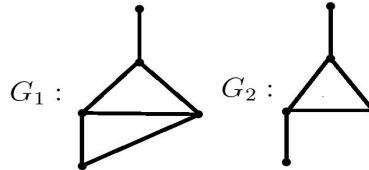

Figure 2.

Fact 1 *If v is a vertex of minimum degree in NNI graph, then at least two of the nonadjacent vertices of v are adjacent themselves and will have the same degree.*

Fact 2 $\overline{K_{p,q}}$ *is NNI if and only if $p \neq q$.*

Fact 3 *If a graph G is NNI, then $\overline{G}$ is not NNI.*
Proof. By Fact 1, there exist at least two nonadjacent vertices of vertex v which are adjacent themselves and of same degree i in G. Those vertices are then nonadjacent vertices and also of same degree $n-1-i$ in $\overline{G}$.

The converse of the Fact 3 is not true. For example, neither the graph P_4 nor its complement is NNI.

Fact 4 *NI graph and NNI graph are isomorphic if and only if graph is self-complementary.*

Fact 5 *Every tree with $n \geq 3$ vertices is not NNI.*

The following theorem gives the construction of connected NNI graph.

Theorem 2.1 *Given a positive integer n and a partition $\{n_1, n_2, ..., n_k\}$ of n such that all n_i's are distinct and $n_i \neq 1$, there exists a connected NNI graph of order $n-k+1$ and size*

$$\frac{1}{2}[n_1^2 + n_2^2 + ---+ n_k^2 - n].$$

Proof. The required connected NNI graph is constructed as follows. The n vertices are partitioned into k sets. The first set consists of n_1 vertices $u_1, u_2, ..., u_{n_1}$, second set consists of n_2 vertices $v_1, v_2, ..., v_{n_2}$ and so on and finally the kth set consists of n_k vertices $z_1, z_2, ..., z_{n_k}$. Then every vertex in the first set is joined to all other vertices of the same set. Similarly each in the remaining sets are joined to all those vertices in the same set. Since each $n_i's$ are distinct and $n_i \neq 1$, we get k distinct complete graphs of order $n_i \geq 2$. From these complete graphs one of them has the maximum order, say n_r. Take one vertex from each remaining $k-1$ complete graphs and then identification of these vertices with different vertices of complete graph n_r gives the connected NNI graph and is denoted

by $NNI_{(n_1,n_2,...,n_k)}$. Therefore the order of NNI graph is $n-k+1$. Since each $n_i's$ forms complete graph and degree of vertex in each n_i is n_i-1.

The size of NNI $=\dfrac{1}{2}\sum degv$

$$=\frac{1}{2}\sum_{i=1}^{k} n_i(n_i-1)$$

$$=\frac{1}{2}[n_1^2+n_2^2+---+n_k^2-n].$$

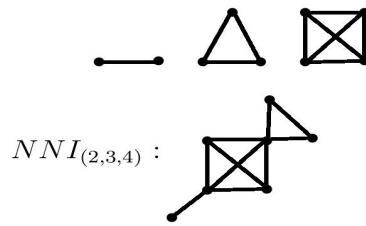Fig. 3 illustrate the construction of connected NNI graph for $n=9$ and the partition $(2,3,4)$.

$$Figure\ 3.$$

Order of $NNI_{(2,3,4)}=n-k+1=9-3+1=7$

Size of $NNI_{(2,3,4)}=\dfrac{1}{2}[4+9+16-9]=10$.

Corollary 2.2 *The maximum size of such a connected NNI graph of order* $n-k+1$ *is* $\dfrac{1}{2}[n_1^2+n_2^2+---+n_k^2-n]$, *where* $n_1^2+n_2^2+---+n_k^2$ *is maximum.*

Corollary 2.3 $n_1^2+n_2^2+---+n_k^2$ *is maximum if and only if* $n_1=n-2$.

Next theorem gives the construction of disconnected NNI graph.

Theorem 2.4 *Given a positive integer* n *and a partition* $\{n_1,n_2,...,n_k\}$ *of* n *such that all* $n_i's$ *are distinct, there exists a disconnected NNI graph of order* n *and size* $\dfrac{1}{2}[n_1^2+n_2^2+---+n_k^2-n]$.

Proof. The required disconnected NNI graph is constructed as follows. The n vertices are partitioned into k sets. The first set consists of n_1 vertices $u_1,u_2,...,u_{n_1}$, second set consists of n_2 vertices $v_1,v_2,...,v_{n_2}$ and so on and finally the kth set consists of n_k vertices $z_1,z_2,...,z_{n_k}$. Then every vertex in the first set is joined to all other vertices of the same set. Similarly each in the remaining sets are joined to all those vertices in the same set. The vertices in the other set are nonadjacent. Therefore, degree of each vertex in the ith set is n_i-1. As all the $n_i's$ are distinct, the graph so constructed is NNI and is denoted by $NNI_{(n_1,n_2,...,n_k)}$. Therefore the order of NNI is $n_1+n_2+---+n_k=n$.

The size of this graph $=\dfrac{1}{2}\sum degv$

$$=\frac{1}{2}\sum_{i=1}^{k} n_i(n_i-1)$$

$$=\frac{1}{2}[n_1^2+n_2^2+---+n_k^2-n].$$

Fig. 4 illustrate the construction of disconnected NNI graph for $n = 8$ and the partition $(1,3,4)$.
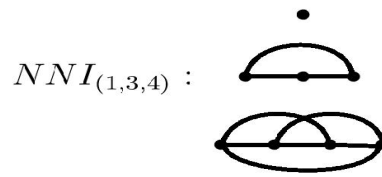
$$NNI_{(1,3,4)} :$$

Figure 4.

Order of $NNI_{(1,3,4)} = 1 + 3 + 4 = 8$

Size of $NNI_{(1,3,4)} = \frac{1}{2}[1 + 9 + 16 - 8] = 9$.

Corollary 2.5 *The maximum size of such a disconnected NNI graph of order n is* $\frac{1}{2}[n_1^2 + n_2^2 + - - - + n_k^2 - n]$, *where* $n_1^2 + n_2^2 + - - - + n_k^2$ *is maximum.*

Corollary 2.6 $n_1^2 + n_2^2 + - - - + n_k^2$ *is maximum if and only if* $n_1 = n - 1$.

3 Properties of NNI graphs

1. Clique graphs.

A *clique* of a graph is a maximal complete subgraph of G. The *clique graph* of G is the intersection graph of all cliques of G.

Consider the connected NNI graph represented by k sets of distinct partitions of n. By construction of this connected NNI graph each clique of it is of order n_i. Also if n_r is the maximum number in the partition of n, then the intersection of the clique of order n_r and the remaining $k - 1$ cliques is nonempty. This shows that the clique graph is a star graph of order k and size $k - 1$. Summarizing, therefore, we get

Theorem 3.1 *Let n be the positive integer with k distinct partition. Then clique graph of connected NNI graph on $n - k + 1$ vertices is a star graph of order k and size $k - 1$.*

Corollary 3.2 *Let n be the positive integer with k distinct partition. Then clique graph of disconnected NNI graph on n vertices is a totally disconnected graph with k vertices.*

2. Vertex covering number.

A set of vertices which covers all the edges of a graph G is called a *vertex cover* of G. The smallest number of vertices in any vertex cover for G is called *vertex covering number* and it is denoted by $\alpha_0(G)$ or α_0.

Theorem 3.3 *The vertex covering number of connected $NNI_{(n_1,n_2,...,n_k)}$ graph with $n_i \neq 1$ is* $n - 2k + 1$.

Proof. Let n_r be maximum number among n_i. Since vertex covering number of complete graph K_{n_i} is $n_i - 1$, then the vertex covering number of connected $NNI_{(n_1,n_2,...,n_k)}$ graph is the sum of the vertex covering number of each complete graph $K_{n_i} (i \neq r)$ and vertex covering number of K_{n_r-k+1}.

Therefore $\alpha_0(NNI_{(n_1,n_2,\ldots,n_k)}) = \sum_{i=1}^{k \neq r}(n_i-1)+n_r-k = [\sum_{i=1}^{k}(n_i-1)-(n_r-1)]+n_r-k$

$$= n-2k+1$$

Corollary 3.4 *The vertex covering number of disconnected* $NNI_{(n_1,n_2,\ldots,n_k)}$ *graph is* $\sum_{i=1}^{k}[n_i-1]=n-k$.

3. Edge covering number.

A set of edges which covers all the vertices of a graph G is called a *edge cover*. The smallest number of edges in any edge cover of G is called its *edge covering number* and it is denoted by $\alpha_1(G)$ or α_1.

Theorem 3.5 *The edge covering number of connected* $NNI_{(n_1,n_2,\ldots,n_k)}$ *graph is*

$\sum_{i=1}^{k}(\lfloor \frac{n_i+1}{2} \rfloor)-\lfloor \frac{n_r+1}{2} \rfloor+\lfloor \frac{n_r-k+2}{2} \rfloor$, *where* n_r *is the maximum number among* n_i *and* $n_i \neq 1$.

Proof. Let n_r be maximum number among n_i. Since edge covering number of complete graph K_{n_i}

is $\lfloor \frac{n_i+1}{2} \rfloor$, then the edge covering number of connected $NNI_{(n_1,n_2,\ldots,n_k)}$ graph is the sum of the edge

covering number of each complete graph K_{n_i} $(i \neq r)$ and edge covering number of K_{n_r-k+1}.

Therefore $\alpha_1(NNI_{(n_1,n_2,\ldots,n_k)}) = \sum_{i=1}^{k \neq r}(\lfloor \frac{n_i+1}{2} \rfloor)+\lfloor \frac{n_r-k+2}{2} \rfloor$

$$= \sum_{i=1}^{k}(\lfloor \frac{n_i+1}{2} \rfloor)-\lfloor \frac{n_r+1}{2} \rfloor+\lfloor \frac{n_r-k+2}{2} \rfloor, \quad \text{where} \quad n_r \quad \text{is the maximum}$$

number among n_i and $n_i \neq 1$.

Corollary 3.6 *The edge covering number of disconnected* $NNI_{(n_1,n_2,\ldots,n_k)}$ *graph with* $n_i \neq 1$ *is*

$\sum_{i=1}^{k}\lfloor \frac{n_i+1}{2} \rfloor$.

4. Vertex independence number.

The set of vertices in G is independent if no two of them are adjacent. The largest number of vertices in such a set is called *vertex independence number* of G and is denoted by $\beta_0(G)$ or β_0.

Theorem 3.7 *The vertex independence number of connected* $NNI_{(n_1,n_2,\ldots,n_k)}$ *graph is* k.

Proof. From Theorem 2.1, order of connected $NNI_{(n_1,n_2,\ldots,n_k)}$ graph is $n-k+1$. Also from Theorem 3.3, $\alpha_0(NNI_{(n_1,n_2,\ldots,n_k)})=n-2k+1$. It is shown in [Ref. [4], p. 95] that for any nontrivial connected (p,q) graph G, $\alpha_0 + \beta_0 = p$.

Therefore $\beta_0(NNI_{(n_1,n_2,\ldots,n_k)})=k$.

5. Edge independence number.

The set of edges in G is independence if no two of them are adjacent. The largest number of edges in such a set is called *edge independence number* of G and is denoted by $\beta_1(G)$ or β_1.

Theorem 3.8 *The edge independence number of connected* $NNI_{(n_1,n_2,...,n_k)}$ *graph is*

$n-k+1-\sum_{i=1}^{k}(\lfloor\frac{n_i+1}{2}\rfloor)+\lfloor\frac{n_r+1}{2}\rfloor-\lfloor\frac{n_r-k+2}{2}\rfloor$, where n_r is the maximum number among n_i and

$n_i \neq 1$.

Proof. From Theorem 2.1, order of connected $NNI_{(n_1,n_2,...,n_k)}$ graph is $n-k+1$. Also from Theorem

3.5, $\alpha_1(NNI_{(n_1,n_2,...,n_k)})=\sum_{i=1}^{k}(\lfloor\frac{n_i+1}{2}\rfloor)-\lfloor\frac{n_r+1}{2}\rfloor+\lfloor\frac{n_r-k+2}{2}\rfloor$, where n_r is the maximum number

among n_i and $n_i \neq 1$. It is shown in [Ref. [4], p. 95] that for any nontrivial connected (p,q) graph

G, $\alpha_1+\beta_1=p$.

Therefore $\beta_1(NNI_{(n_1,n_2,...,n_k)})=n-k+1-\sum_{i=1}^{k}(\lfloor\frac{n_i+1}{2}\rfloor)+\lfloor\frac{n_r+1}{2}\rfloor-\lfloor\frac{n_r-k+2}{2}\rfloor$, where n_r is the

maximum number among n_i and $n_i \neq 1$.

6. Domination number.

A set D of vertices in a graph G is called a *dominating set* if every vertex in $V-D$ is adjacent to some vertex in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G.

Consider the connected $NNI_{(n_1,n_2,...,n_k)}$ graph represented by k sets of distinct partitions of n. By construction of this connected $NNI_{(n_1,n_2,...,n_k)}$ graph if n_r is the maximum number in the partition of n, then taking one vertex from each remaining $k-1$ sets we get a dominating set, which has a minimum cardinality. This shows that the domination number of connected $NNI_{(n_1,n_2,...,n_k)}$ graph is $k-1$.

Summarizing, therefore, we get

Theorem 3.9 *The domination number of connected* $NNI_{(n_1,n_2,...,n_k)}$ *graph is* $k-1$.

Corollary 3.10 *The domination number of disconnected* $NNI_{(n_1,n_2,...,n_k)}$ *graph is* k.

Open problem.
Given a positive integer n, can we characterize all those NNI graphs of order n?

The classes of all $NNI_{(n_1,n_2,...,n_k)}$ graph described in this paper is only a proper subclasses of the class of all NNI graphs. For example, the following graph shown in Fig. 5 is the NNI graph of order 5; but it does not come under the purview of any $NNI_{(n_1,n_2,...,n_k)}$ graph of order 5.
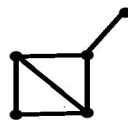

Figure 5.

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